Lecture 15. Graph colouring II

Yanbo ZHANG

Hebei Normal University

Planar graph is 6-colourable

Claim

A (simple) planar graph G contains a vertex v of degree at most 5.

Proof.

Recall that in a planar graph $|E(G)| \leq 3|V| - 6$. Thus, we have that $\sum_{v \in V(G)} d(v) \leq 6|V| - 12 < 6|V|$ and so the claim follows. □

Corollary

A planar graph G is 5-degenerate and thus 6-colourable.

Planar graph is 5-colourable (5 colour theorem)

Theorem (5 colour theorem; Heawood 1890)

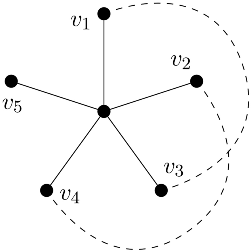
Every planar graph G is 5-colourable.

Proof.

By induction of $|V(G)|$. For $|V(G)| \le 5$ the statement is obvious. Assume $|V(G)| > 5$. Let v be a vertex of degree at most 5 in G. By induction, $G \setminus v$ is 5-colourable. If $d(v) < 5$, then a colouring of $f : V(G) \setminus \{v\} \to \{1, \ldots, 5\}$ can be extended to $V(G)$ by assigning $f(v) \in \{1, \ldots, 5\} \setminus \{f(u) : uv \in E(G)\}$. Hence, we may assume that $d(v) = 5$.

Proof.

Fix a planar embedding of G in which the neighbours of v are coloured by f with the colours $1,\ldots,5$ in clockwise order (if f uses less than 5 colours on $N(v)$ then it can be extended to $V(G)$ as before). Let the corresponding vertices be $v_1,\ldots,v_5$, i.e., $f(v_i)=i$ for $i=1,\ldots,5$.



Planar graph is 5-colourable (5 colour theorem)

Proof.

For $1 \leq i \neq j \leq 5$ let G_{ij} be the subgraph of $G \setminus v$ induced by the colours i and j. Switching the two colours in any connected component of G_{ij} again gives a proper 5-colouring of $G \setminus v$. If the component of G_{ij} containing v_i does not contain v_j then we switch the colours in the component of G_{ij} which contains v_i, in order to remove the colour i from $N(v)$; we can then colour v with the colour i.

Proof.

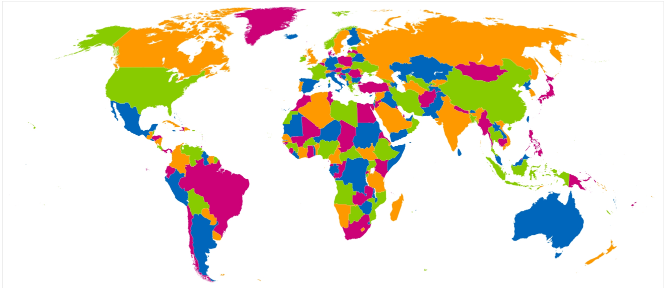
We can therefore assume that for every pair $1 \leq i \neq j \leq 5$ the component of G_{ij} containing v_i also contains v_j. Let P_{ij} be a path in G_{ij} from v_i to v_j. Obviously, the vertices of P_{ij} are coloured alternatively by colours i and j. Consider paths P_{13} and P_{24}. By the Jordan curve theorem, they should intersect. Since the drawing is planar, they intersect in a vertex. But all the vertices of P_{13} are coloured 1 and 3 and all the vertices of P_{24} are coloured 2 and 4, contradiction! □

Planar graph is 4-colourable (4 colour theorem)

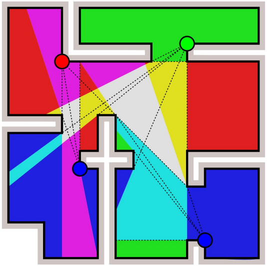
Theorem (Appel-Haken 1977; conjectured by Guthrie in 1852)

Every planar graph is 4-colourable. (the countries of every plane map can be 4-coloured so that neighbouring countries get distinct colours).

Remark. The only known proofs heavily use computers.

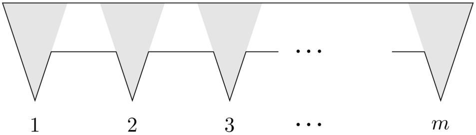


Here is an appealing problem which was raised by Victor Klee in 1973. Suppose the manager of a museum wants to make sure that at all times every point of the museum is watched by a guard. The guards are stationed at fixed posts, but they are able to turn around. How many guards are needed?



The art gallery problem

We picture the walls of the museum as a polygon consisting of n sides. Of course, if the polygon is convex, then one guard is enough. In fact, the guard may be stationed at any point of the museum. But, in general, the walls of the museum may have the shape of any closed polygon.



Consider a comb-shaped museum with $n = 3m$ walls, as depicted above. It is easy to see that this requires at least $m = n/3$ guards. In fact, there are n walls. Now notice that the point 1 can only be observed by a guard stationed in the shaded triangle containing 1, and similarly for the other points $2, 3, \ldots, m$. Since all these triangles are disjoint we conclude that at least m guards are needed. But m guards are also enough, since they can be placed at the top lines of the triangles. By cutting off one or two walls at the end, we conclude that for any n there is an n-walled museum which requires $\lfloor n/3 \rfloor$ guards.

The following result states that $\lfloor n/3 \rfloor$ is the worst case.

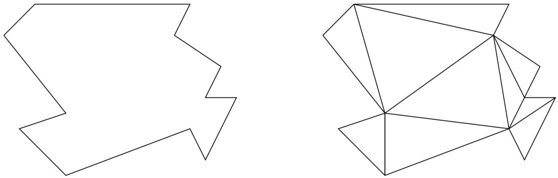
Theorem

For any museum with n walls, $\lfloor n/3 \rfloor$ guards suffice.

This "art gallery theorem" was first proved by Vašek Chvátal by a clever argument, but here is a proof due to Steve Fisk that is truly beautiful.

Proof.

First of all, let us draw $n-3$ non-crossing diagonals between corners of the walls until the interior is triangulated. For example, we can draw 9 diagonals in the museum depicted below to produce a triangulation. It does not matter which triangulation we choose, any one will do.



A museum with 12 walls, and a triangulation of it

The art gallery theorem

Now think of the new figure as a plane graph with the corners as vertices and the walls and diagonals as edges.

Claim

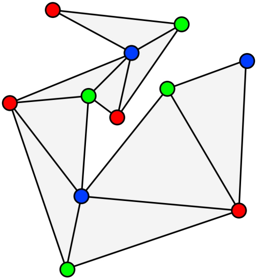
This graph is 3-colorable.

Proof.

For $n = 3$ there is nothing to prove. Now for $n > 3$ pick any two vertices u and v which are connected by a diagonal. This diagonal will split the graph into two smaller triangulated graphs both containing the edge uv. By induction we may color each part with 3 colors where we may choose color 1 for u and color 2 for v in each coloring. Pasting the colorings together yields a 3-coloring of the whole graph. □

Proof.

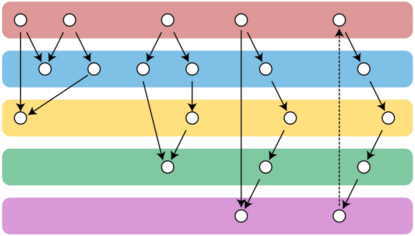
The rest is easy. Since there are n vertices, at least one of the color classes, say the vertices colored 1, contains at most $\lfloor n/3 \rfloor$ vertices, and this is where we place the guards. Since every triangle contains a vertex of color 1 we infer that every triangle is guarded, and hence so is the whole museum. □



Gallai-Roy theorem

Theorem (Gallai-Roy theorem)

If D is an orientation of G with longest path length $\ell(D)$, then $\chi(G) \leq 1 + \ell(D)$. Furthermore, equality holds for some orientation of G.



Proof.

Suppose D is an orientation of G. Let D' be a maximal acyclic subdigraph of D (this means that adding any additional edge of D to D' would create a directed cycle). An obvious way to obtain such a subgraph is to start from the empty digraph on the same set of vertices, and arbitrarily add edges of D one-by-one as long as these new edges do not create a directed cycle. When we cannot add any further edges, we will have a suitable subdigraph D'.

Proof.

Colour $V(G)$ by letting $f(v)$ be one more than the length of the longest path in D' that ends at v. Because D' is acyclic, for every edge $u \to v$, the longest path in D' ending in u cannot contain v. Therefore any path ending at u can be extended to a longer path ending at v, showing that $f(v) > f(u)$. This shows that f strictly increases along each path in D'.

Gallai-Roy theorem

Proof.

The colouring f uses colours 1 through $1+\ell(D')$ on $V(D')=V(G)$. We now prove that f is a proper colouring of G. If $(u,v)\in E(D')$, the above discussion immediately shows that $f(u)\neq f(v)$. Otherwise, if $(u,v)\in E(D)$ which is not in D', there is a path in D' between its endpoints (since the addition of (u,v) to D' creates a cycle). This again implies $f(u)\neq f(v)$, since f increases along paths of D'.

Proof.

To prove the second statement, we construct an orientation D^* such that $\ell(D^*) \le \chi(G) - 1$. Let f be an optimal colouring of G. For each edge (u,v) in G, orient the edge as $u \to v$ in D^* if and only if $f(u) < f(v)$. Since f is a proper colouring, this defines an orientation. Since the labels used by f increase along each path in D^*, and there are only $\chi(G)$ labels in f, we have $\ell(D^*) \le \chi(G) - 1$. □

Thank you!