Lecture 14. Graph colouring I

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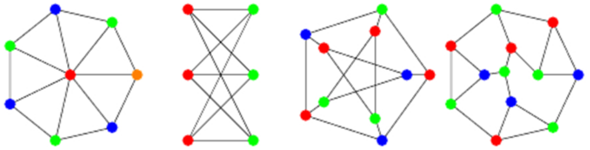
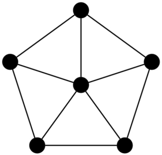
Vertex colouring

Definition

A k-colouring of G is a labeling $f\colon V(G) \to \{1,\ldots,k\}$. It is a proper k-colouring if $(x,y) \in E(G)$ implies $f(x) \neq f(y)$. A graph G is k-colourable if it has a proper k-colouring. The chromatic number $\chi(G)$ is the minimum k such that G is k-colourable. If $\chi(G) = k$, then G is k-chromatic. If $\chi(G) = k$, but $\chi(H) < k$ for every proper subgraph H of G, then G is colour-critical or k-critical.

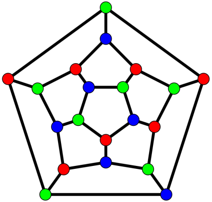


Example.

1. $\chi(K_n) = n$.
2. The chromatic number of an odd cycle is 3.
3. $G =$  , $\chi(G) = 4$.

Remark

Remark. The vertices having a given colour in a proper colouring must form an independent set, so $\chi(G)$ is the minimum number of independent sets needed to cover $V(G)$. Hence G is k-colourable if and only if G is k-partite. Multiple edges do not affect chromatic number. Although we define k-colouring using numbers from $\{1, \ldots, k\}$ as labels, the numerical values are usually unimportant, and we may use any set of size k as labels.



Example. (examination scheduling). The students at a certain university have annual examinations in all the courses they take. Naturally, examinations in different courses cannot be held concurrently if the courses have students in common. How can all the examinations be organized in as few parallel sessions as possible? To find a schedule, consider the graph G whose vertex set is the set of all courses, two courses being joined by an edge if they give rise to a conflict. Clearly, independent sets of G correspond to conflict-free groups of courses. Thus, the required minimum number of parallel sessions is the chromatic number of G.

Example. (chemical storage). A company manufactures n chemicals $C_1, C_2, \dots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure, the company wishes to divide its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned? We obtain a graph G on the vertex set $\{v_1, v_2, \dots, v_n\}$ by joining two vertices v_i and v_j if and only if the chemicals C_i and C_j are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of G.

Simple bounds on the chromatic number

Claim

If H is a subgraph of G then $\chi(H) \le \chi(G)$.

Proof.

Note that a proper colouring of G is also a proper colouring of H. □

Simple bounds on the chromatic number

Claim

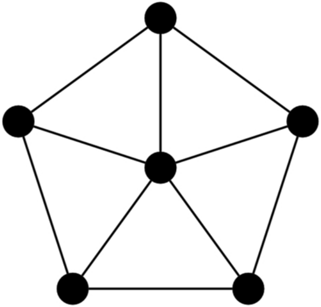
$\chi(G) \geq \omega(G)$.

Proof.

Let $\omega(G) = t$. Then G contains a subgraph H which is isomorphic to K_t. Thus, by the claim above it follows that $\chi(G) \geq \chi(H) = t$. □

Example

Consider the following graph.



In this case we have $\chi(G) = 4$ and $\omega(G) = 3$. Thus, the chromatic number can be bigger than the clique number.

Simple bounds on the chromatic number

Proposition

$\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$.

Proof.

Let $\chi(G) = k$. A k-colouring of $V(G)$ gives a partition $V(G) = V_1 \cup \ldots \cup V_k$ such that every V_i is an independent set. Hence, $|V_i| \leq \alpha(G)$. Therefore, $|V(G)| = \sum_{i=1}^{k} |V_i| \leq k\alpha(G)$. Thus, $k = \chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$ as claimed. □

Simple bounds on the chromatic number

Claim

For any graph $G = (V, E)$ *and any* $U \subseteq V$ *we have* $\chi(G) \leq \chi(G[U]) + \chi(G[V \setminus U])$.

Proof.

Properly colour U using $\chi(G[U])$ colours and properly colour $V \setminus U$ using $\chi(G[V \setminus U])$ other colours. This gives a proper colouring of G in $\chi(G[U]) + \chi(G[V \setminus U])$ colours. □

Simple bounds on the chromatic number

Claim

For any graphs G_1 and G_2 on the same vertex set, $\chi(G_1 \cup G_2) \le \chi(G_1)\chi(G_2)$.

Proof.

Let c_1 and c_2 be colourings of G_1 and G_2 with the integers in $[\chi(G_1)]$ and $[\chi(G_2)]$ respectively. We colour the vertices of $G_1 \cup G_2$ with elements of the set $[\chi(G_1)] \times [\chi(G_2)]$, with the colouring c defined by $c(v) = (c_1(v), c_2(v))$. If v is adjacent to w in $G_1 \cup G_2$ then (v,w) is an edge in one of G_1 or G_2, so $c_1(v) \ne c_1(w)$ or $c_2(v) \ne c_2(w)$. This proves that $c(v) \ne c(w)$, so c is proper. □

Proposition

(i) $\chi(G)\chi(\overline{G}) \geq |G|$.

Proof.

(i) follows from the last claim: we have

$\chi(G)\chi(\overline{G}) \geq \chi(G \cup \overline{G}) = \chi(K_{|G|}) = |G|$. □

Simple bounds on the chromatic number

Proposition

(ii) $\chi(G) + \chi(\overline{G}) \leq |G| + 1$.

Proof.

(ii) can be proved by induction on $|G|$ (the case $|G| = 1$ is obvious). So, let $|G| = n + 1$. Let $G_0 = G \backslash v$ for some vertex v. By induction we have $\chi(G_0) + \chi(\overline{G_0}) \leq n + 1$. Let $c : V \to [k]$ be a colouring of G_0 and $f : V \to [\ell]$ be a colouring of $\overline{G_0}$, with $k + \ell = n + 1$ (we might be using more colours than are necessary).

Simple bounds on the chromatic number

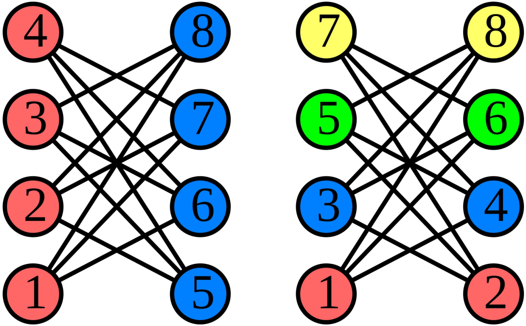
Proof.

If $d_G(v) < k$ then there is a colour c_v such that v has no neighbours coloured c_v. We can then colour v with c_v to extend c to a colouring of G with k colours. This would prove $\chi(G) \leq k$, and since $\chi(\overline{G}) = \chi(\overline{G_0} \cup \{v\}) \leq \ell + 1$ we have $\chi(G) + \chi(\overline{G}) \leq k + \ell + 1 \leq n + 2$. Otherwise $d_G(v) \geq k$ so $d_{\overline{G}}(v) \leq n - k = \ell - 1$. We can then use exactly the same reasoning as before to extend f to a colouring of $\overline{G}$ with ℓ colours, and since $\chi(G) \leq k + 1$ we are done again. □

Greedy colouring

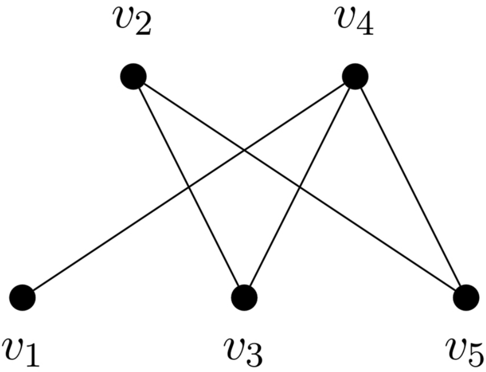
Definition

The greedy colouring with respect to a vertex ordering $v_1, \ldots, v_n$ of $V(G)$ is obtained by colouring vertices in the order $v_1, \ldots, v_n$, assigning to v_i the smallest-indexed colour not already used on its lower-indexed neighbours.



Example

This graph has chromatic number 2 but the greedy colouring needs 3 colours.



k-degenerate

Definition

Let $G = (V, E)$ be a graph. We say that G is k-degenerate if every subgraph of G has a vertex of degree less than or equal to k.

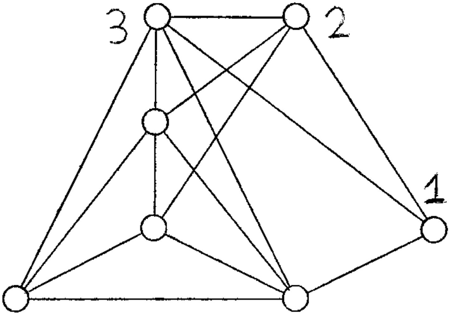


Figure: 3-degenerate graph

k-degenerate

Proposition

G is k-degenerate if and only if there is an ordering $v_1,\ldots,v_n$ of the vertices of G such that each v_i has at most k neighbours among the vertices $v_1,\ldots,v_{i-1}$.

Proof.

If there is such an ordering, then for any subgraph H, consider the maximum vertex of H with respect to the ordering. This vertex has at most k neighbours in H, thus proving that G is k-degenerate.

Proof.

Conversely, suppose G is k-degenerate. We prove the existence of a suitable ordering by induction on the number of vertices. If G is k-degenerate it has a vertex of degree at most k. Call this vertex v_n. Let $G' = G \backslash v_n$ and note that G' is still k-degenerate. Thus, there exists an ordering $v_1, \ldots, v_{n-1}$ of the vertices of G' satisfying the assertion of the proposition for G'. Then the ordering $v_1, \ldots, v_n$ satisfies the required conditions for G. □

Definition

Define $\mathrm{dg}(G)$ to be the minimum k such that G is k-degenerate.

Remark. $\delta(G) \leq \mathrm{dg}(G) \leq \Delta(G)$.

k-degenerate

Theorem

$\chi(G) \le 1 + \mathrm{dg}(G)$.

Proof.

Let $k = \mathrm{dg}(G)$. Fix an ordering $v_1, \ldots, v_n$ of $V(G)$ such that each v_i has at most k neighbours among $v_1, \ldots, v_{i-1}$. Use the greedy colouring on G with respect to this vertex ordering. This colouring uses at most $k+1$ colours, because when one colours v_i there are at most k colours which cannot be used. □

Corollary

$\chi(G) \leq \Delta(G) + 1$.

Note that $\text{dg}(G)$ can be much smaller than $\Delta(G)$, for example if $G = K_{3,n-3}$ we have $\text{dg}(G) = 3$ but $\Delta(G) = n - 3$.

Remark. This bound is tight if $G = K_n$ or if G is an odd cycle.

Brooks' theorem

For any simple graph G, the chromatic number $\chi(G) \leq \Delta(G) + 1$.

Theorem (Brooks 1941)

If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

If G is a clique or an odd cycle, then $\chi(G) = \Delta(G) + 1$.

Brooks' theorem

We will present a recent new proof due to Mariusz Zając which apart from being self contained and simpler than previous proofs has the advantage of being easily converted into an algorithm. The idea of the proof is to use induction on the number of vertices, however property of being connected is not very amenable to such arguments. The key idea of this proof is to show a slightly stronger result, which replaces the connectedness condition with that of not having a clique of certain size as a subgraph.

Theorem (Zając's result 2018)

Let $k \geq 3$ be a natural number. Let G be a graph with $\triangle(G) \leq k$. If G does not contain a clique on $k+1$ vertices, then G is k-colorable.

Brooks' theorem

Theorem (Brooks 1941)

If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Proof.

If $\Delta(G) = 1$ the graph can only be a single edge, which is a clique. If $\Delta(G) = 2$ the graph is a union of disjoint paths and cycles and since it is connected it is a path or a cycle. For paths and even cycles by colouring vertices alternately it is easy to see they are 2-colourable. Let now $k = \Delta(G) \geq 3$, the above theorem implies that G can be k-coloured unless it contains K_{k+1} as a subgraph. But since $\Delta(G) = k$ this clique can send no edges to the rest of the graph, so since G is connected $G = K_{k+1}$. □

Zając's result

Theorem (Zając's result 2018)

Let $k \geq 3$ be a natural number. Let G be a graph with $\triangle(G) \leq k$. If G does not contain a clique on $k+1$ vertices, then G is k-colorable.

Proof.

We will make use of the following easy observation. Suppose that G is partially coloured using at most k colours. Let $P = v_1 v_2 \dots v_j$ be a path in G, and assume that the vertices of P are uncoloured. Then we may colour all vertices from v_1 up to v_{j-1} consecutively along P, since at the moment of colouring the vertex v_i its neighbour v_{i+1} is yet uncoloured, so v_i has at most $k-1$ coloured neighbours. We denote this sequential colouring procedure by $\textsc{PathColour}\,(v_1, v_2, \dots, v_{j-1}; v_j)$.

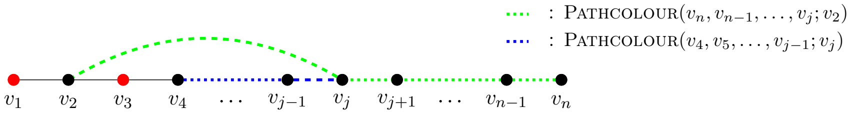
Proof.

Note that after its execution the last vertex v_j of the path P remains uncoloured, in particular PATHCOLOUR does nothing if $j = 1$.

The proof now proceeds by induction on the number n of vertices of G. For $n \le k$ the assertion holds trivially. We may further assume that G is k-regular, since otherwise we would delete a vertex of degree strictly less than k and apply induction. Let v be any vertex of G. Since G does not contain a clique on $k+1$ vertices, there exist two neighbours x, y of v that are not adjacent in G. Denote $v_1 = x, v_2 = v$, and $v_3 = y$. Let $P = v_1 v_2 v_3 \dots v_r$ be a path starting with these three vertices and extending itself maximally, i.e. until some vertex v_r whose all neighbours are already on P.

Proof.

Case 1. Suppose first that $r = n$, which means that P contains all vertices of G, and let v_j be any neighbour of v_2 other than v_1 and v_3 (it exists since $k \geq 3$). We start by giving the vertices v_1 and v_3 the same colour. Then apply procedures PATHCOLOUR $(v_4, v_5, \ldots, v_{j-1}; v_j)$ and PATHCOLOUR $(v_n, v_{n-1}, \ldots, v_j; v_2)$. Finally, colour the vertex v_2, which is possible because it has two neighbours in the same colour. The entire graph G is now k-coloured.



Proof.

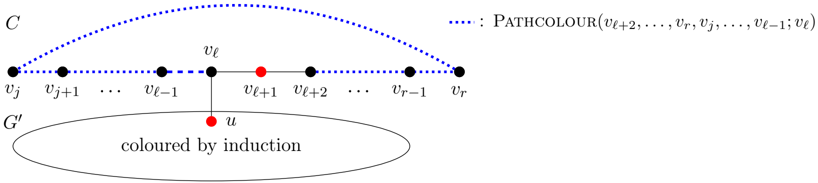
Case 2. Assume now that $r < n$. Recall that all neighbours of v_r are on the path P. Let v_j be the neighbour of v_r with the smallest index. So, $C = v_j v_{j+1} \dots v_r$ is a cycle in G. Consider the subgraph $G' = G - C$ obtained by deleting all vertices of C. We first, colour G' using k colours by the induction hypothesis. If there is no edge between G' and C, then we are done by applying induction also to the subgraph induced by C.

If, on the contrary, there is a vertex on C with a neighbour in G', then let v_ℓ be such vertex with the largest index, and let u be any of its neighbours in G'.

Zając's result

Proof.

Notice that $\ell < r$ because v_r has all of its neighbours on C. Since the vertex $v_{\ell+1}$ does not have neighbours in G', we may assign it the same colour as u. Now apply procedure PathColour $(v_{\ell+2},\dots,v_r,v_j,\dots,v_{\ell-1};v_\ell)$ and finally colour v_ℓ, which is possible as it has two neighbours in the same colour. As previously, the entire graph G is coloured and the proof is complete. □



Thank you!