Lecture 13. Platonic Solids

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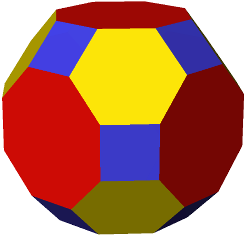
1. Platonic solids
2. Euler's polyhedral formula
3. Platonic solids characterization

Polytope

Definition

A polytope is a solid in 3 dimensions with flat faces, straight edges and sharp corners. Faces of a polytope are joined at the edges. A polytope is convex if the line connecting any two points of the polytope lies inside the polytope.

Example.

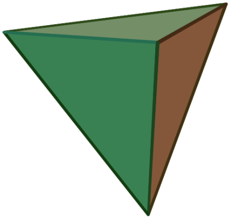


Definition

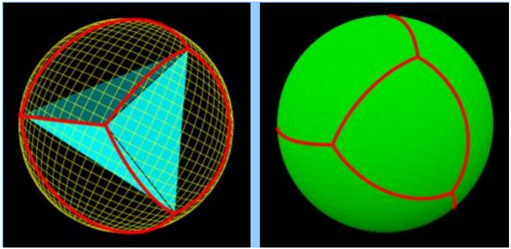
A regular solid or Platonic solid is a convex polytope which satisfies the following:

1. all of its faces are congruent regular polygons,
2. all vertices have the same number of faces adjacent to them.

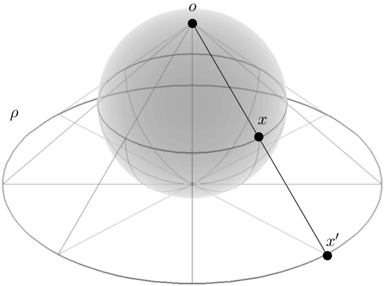
Example. The tetrahedron:



We will now characterise all Platonic solids. The first step is to convert a convex polytope into a planar graph. To do this, we place the considered polytope inside a sphere. Then we project the polytope onto the sphere (imagine that the edges of the polytope are made from wire and we place a tiny lamp in the center). This yields a graph drawn on the sphere without edge crossings.



Now let us show that planar graphs are exactly graphs that can be drawn on the sphere. This becomes quite obvious if we use the stereographic projection. We place the sphere in the 3-dimensional space in such a way that it touches the considered plane ρ. Let o denote the point of the sphere lying farthest from ρ, the 'north pole'.

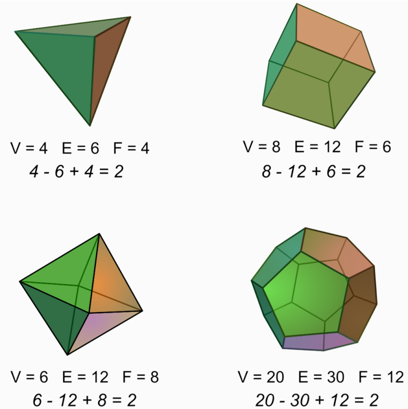


Then the stereographic projection maps each point $x \neq o$ of the sphere to a point x', where x' is the intersection of the line ox with the plane ρ. (For the point o, the projection is undefined.) This defines a bijection between the plane and the sphere without the point o. Given a drawing of a graph G on the sphere without edge crossings, where the point o lies on no arc of the drawing (which we may assume by a suitable choice of o), the stereographic projection yields a planar drawing of G. Conversely, from a planar drawing we get a drawing on the sphere by the inverse projection.

Euler's polyhedral formula

Corollary

If K is a convex polytope with v vertices, e edges and f faces, then $v - e + f = 2$.



Platonic solids characterization

Suppose K is a Platonic solid. All its faces are congruent; assume that they have n vertices (and, thus, n edges). Let us assume moreover that each vertex is adjacent to m faces (and, thus, it has m edges adjacent to it). Since each edge is adjacent to exactly two faces,

$$2e = nf.$$

Moreover, each edge is adjacent to two vertices, and one vertex belongs to m edges, thus

$$mv = 2e.$$

Expressing v and f in terms of e, and substituting to Euler's formula, we obtain that $\frac{2e}{m} - e + \frac{2e}{n} = 2$. Rearranging, we arrive at

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{e}.$$

Note that since K is a 3-dimensional polytope, each of its faces is a polygon and thus has at least 3 vertices; that is, $n \geq 3$. Moreover, at each vertex, there are at least three faces meeting; $m \geq 3$. On the other hand, since $e \geq 1$, we must have

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}.$$

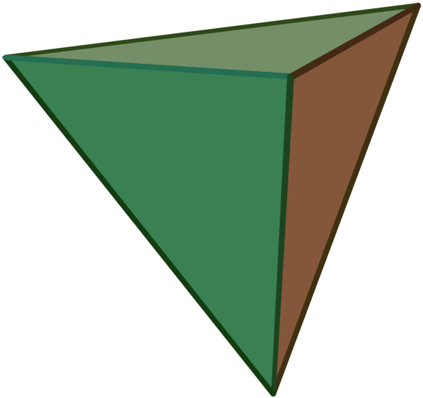
These conditions do not leave too much leeway.

From $n \geq 3$, $m \geq 3$, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ we see that there are only five possible (n, m) pairs for which the above inequality holds. These are $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, $(5, 3)$.

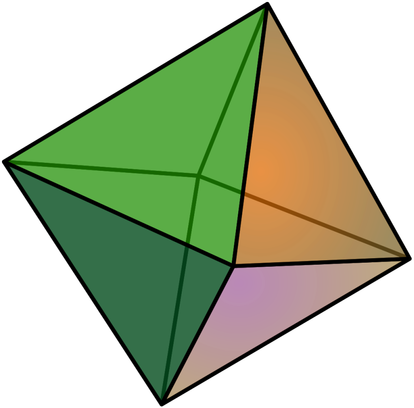
A Platonic solid corresponds to each of these pairs. We list them below.

Platonic solids characterization

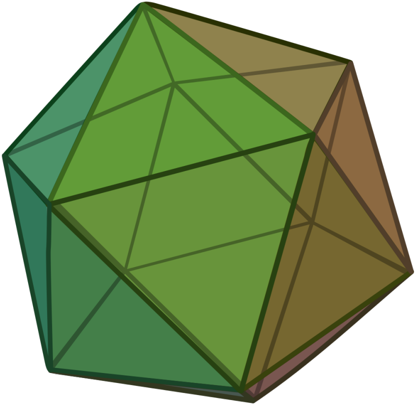
- Tetrahedron. Here $n = 3$ and $m = 3$. Thus, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ yields that $e = 6$. By $mv = 2e$, $v = 4$, and by $2e = nf$, $f = 4$. There are 4 vertices and 4 faces of the tetrahedron; the faces are regular triangles, and the vertices are adjacent to 3 edges.



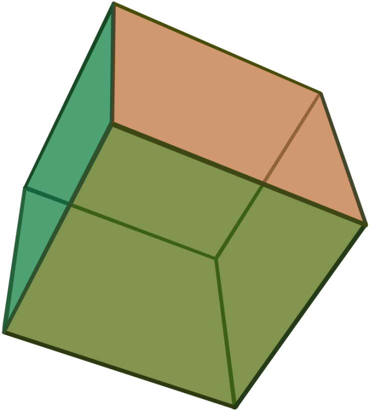
- Octahedron. Here $n = 3$ and $m = 4$. Thus, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ yields that $e = 12$. By $mv = 2e$, $v = 6$, and by $2e = nf$, $f = 8$. There are 8 vertices and 8 faces of the octahedron; the faces are regular triangles, and the vertices are adjacent to 4 edges.



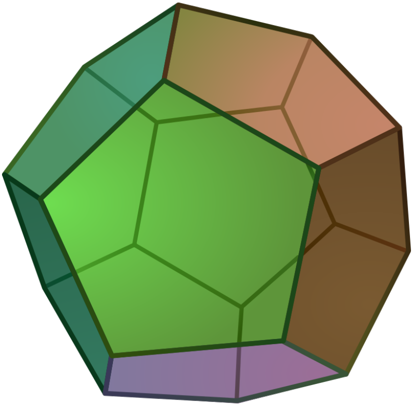
- Icosahedron. Here $n = 3$ and $m = 5$. Thus, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ yields that $e = 30$. By $mv = 2e$, $v = 12$, and by $2e = nf$, $f = 20$. There are 12 vertices and 20 faces of the icosahedron; the faces are regular triangles, and the vertices are adjacent to 5 edges.



- Cube. Here $n = 4$ and $m = 3$. Thus, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ yields that $e = 12$. By $mv = 2e$, $v = 8$, and by $2e = nf$, $f = 6$. There are 8 vertices and 6 faces of the tetrahedron; the faces are squares, and the vertices are adjacent to 3 edges.



- Dodecahedron. Here $n = 5$ and $m = 3$. Thus, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ yields that $e = 30$. By $mv = 2e$, $v = 20$, and by $2e = nf$, $f = 12$. There are 20 vertices and 12 faces of the tetrahedron; the faces are regular pentagons, and the vertices are adjacent to 3 edges.



Thank you!