Lecture 12. Planar graphs

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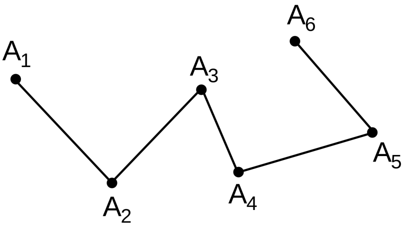
1. Planar graphs

2. Euler's formula

Polygonal path

Definition

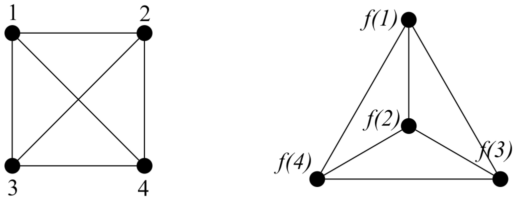
A polygonal path or polygonal curve in the plane is the union of many line segments such that each segment starts at the end of the previous one and no point appears in more than one segment except for common endpoints of consecutive segments. In a polygonal u,v-path, the beginning of the first segment is u and the end of the last segment is v.



Planar graph and plane graph

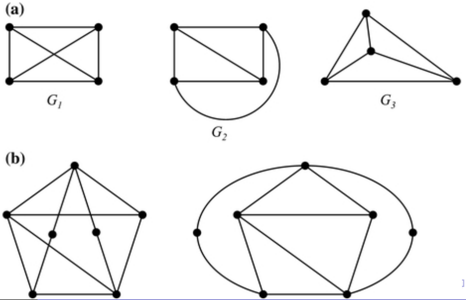
Definition

A drawing of a graph G is a function that maps each vertex $v \in V(G)$ to a point $f(v)$ in the plane and each edge uv to a polygonal $f(u),f(v)$-path in the plane. The images of vertices are distinct. A point in $f(e) \cap f(e')$ other than a common end is a crossing. A graph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G. A plane graph is a particular drawing of a planar graph in the plane with no crossings.



Remark.

1. Even if a graph does not look like it is planar, it may still be drawn without crossings. So it is planar.
2. We get the same class of graphs if we only require images of edges to be continuous curves. This is because any continuous line can be arbitrarily accurately approximated by a polygonal curve.



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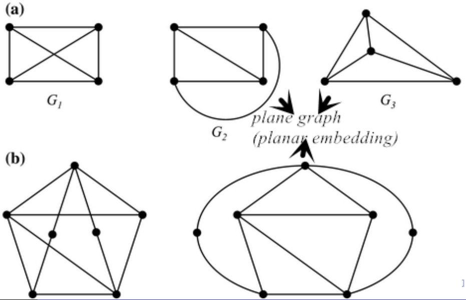
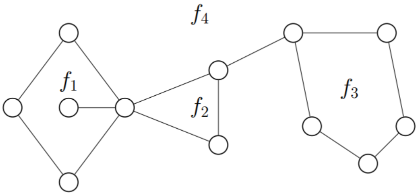


Faces of a plane graph

Definition

An open set in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every $p \in U$, all points within some small distance from p belong to U. A region is an open set U that contains a polygonal u,v-path for every pair $u,v \in U$ (that is, it is "path-connected"). The faces of a plane graph are the maximal regions of the plane that are disjoint from the drawing.



A planar embedding with 4 faces

Faces of a plane graph

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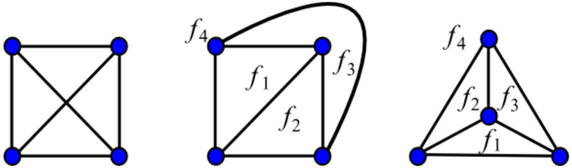
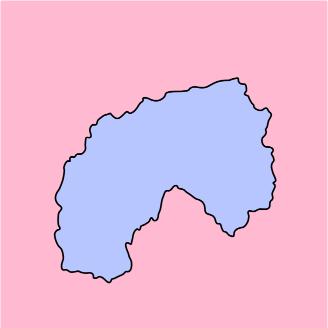


Jordan curve theorem

Theorem (Jordan curve theorem)

A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary.



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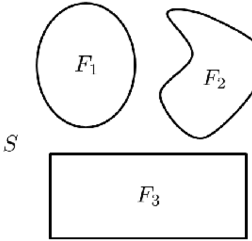
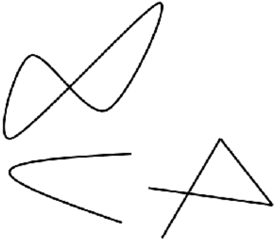
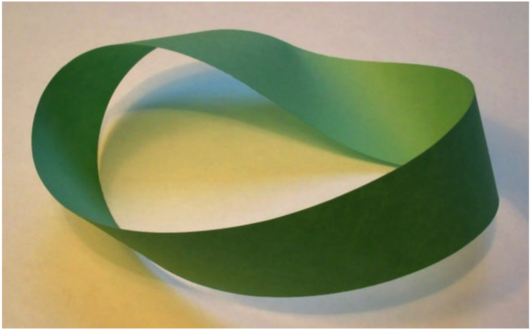


Jordan Curves

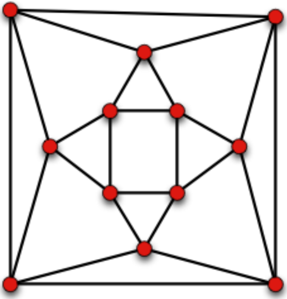


Not Jordan Curves

Remark. This is not true in three dimensions. In $\mathbb{R}^3$ there is a surface called the Möbius band which has only one side.



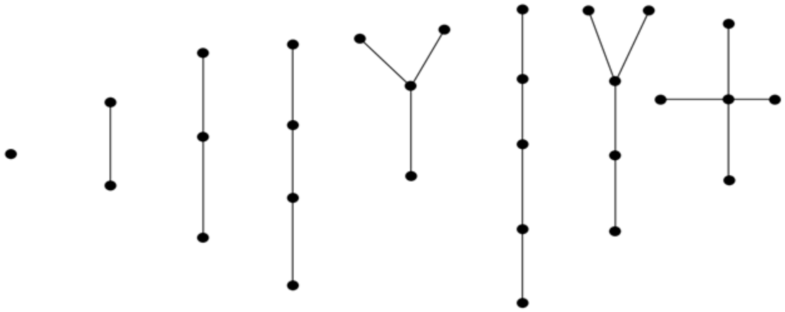
Remark. The faces of a plane graph G are pairwise disjoint (they are separated by the edges of G). Two points are in the same face if and only if there is a polygonal path between them which does not cross an edge of G. Also, note that a finite graph has a single unbounded face (the area "outside" of the graph).



Plane forest

Proposition

A plane forest has exactly one face.



The length of a face

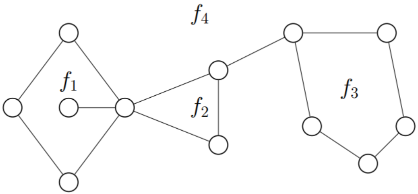
Definition

The length of the face f in a planar embedding of G is the sum of the lengths of the walks in G that bound it.

Example

Let $l(f_i)$ denote the length of a face f_i.

Then $l(f_1) = 6$, $l(f_2) = 3$, $l(f_3) = 5$, $l(f_4) = 14$.



A planar embedding with 4 faces

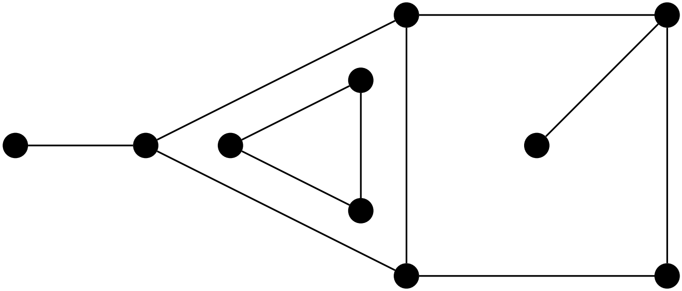
The length of a face

Definition

The length of the face f in a planar embedding of G is the sum of the lengths of the walks in G that bound it.

Example

The following graph has 4 faces of lengths 6, 6, 3 and 7.



Proposition

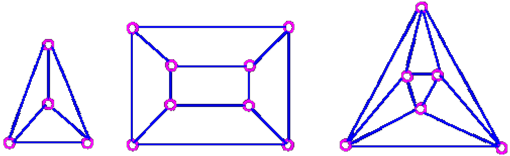
If $l(f_i)$ denotes the length of a face f_i in a plane graph G, then $2e(G) = \sum l(f_i)$.

Proof.

In the sum $\sum l(f_i)$, every edge was counted twice. □

Theorem (Euler's formula 1758)

If a connected plane graph G has exactly n vertices, e edges and f faces, then $n - e + f = 2$.



In the first graph, $n = 4$, $e = 6$, $f = 4$, $n - e + f = 2$.

In the second graph, $n = 8$, $e = 12$, $f = 6$, $n - e + f = 2$.

In the third graph, $n = 6$, $e = 12$, $f = 8$, $n - e + f = 2$.

Theorem (Euler's formula 1758)

If a connected plane graph G has exactly n vertices, e edges and f faces, then $n - e + f = 2$.

Proof.

We use induction on the number of edges in G. If $e(G) = n - 1$ and G is connected, then G is a tree. We have $f = 1, e = n - 1$. Thus $n - e + f = 2$ holds.

If $e(G) \geq n$ and G is connected, G contains a cycle C. Choose any edge g on C. Let $G' = G \setminus g$. Then G' is connected and $e(G') \geq n - 1$. By the inductive hypothesis, for G', we have $n' - e' + f' = 2$. Here $n' = n$ and $e' = e - 1$. Also, deleting g unites two faces, so $f' = f - 1$. Thus,

$$n - e + f = 2.$$

Euler's formula

Remark. The fact that deleting an edge in a cycle decreases the number of faces by one can be proved formally using the Jordan curve theorem.

Euler's formula

Theorem

If G is a simple planar graph with at least three vertices, then $e(G) \leq 3|G| - 6$. *If G is also triangle-free, then* $e(G) \leq 2|G| - 4$.

Proof.

It suffices to consider connected graphs; otherwise we could add edges to connect the graph. Every face boundary in a simple graph contains at least three edges. Let $\{f_i\}$ be the list of face lengths. Then $2e = \sum_{1 \leq i \leq f} f_i \geq 3f$. Hence $f \leq \frac{2}{3}e$. Substitute this into Euler's formula. We have

$$n - e + \frac{2}{3}e \geq n - e + f = 2,$$

thus $n - \frac{1}{3}e \geq 2$ and $e \leq 3n - 6$.

Theorem

If G is a simple planar graph with at least three vertices, then $e(G) \leq 3|G| - 6$. If G is also triangle-free, then $e(G) \leq 2|G| - 4$.

Proof.

When G is triangle-free, the faces have length at least 4 (except in the case of K_2). In this case $2e = \sum f_i \geq 4f$, and we have

$$n - e + \frac{1}{2}e \geq n - e + f = 2,$$

thus $n - \frac{1}{2}e \geq 2$ and $e \leq 2n - 4$. □

Euler's formula

Theorem

If G is a simple planar graph with at least three vertices, then $e(G) \leq 3|G| - 6$. If G is also triangle-free, then $e(G) \leq 2|G| - 4$.

Corollary

If G is a planar bipartite n-vertex graph with $n \geq 3$ vertices then G has at most $2n - 4$ edges.

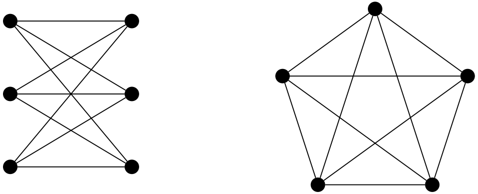
Euler's formula

Corollary

K_5 and $K_{3,3}$ are not planar.

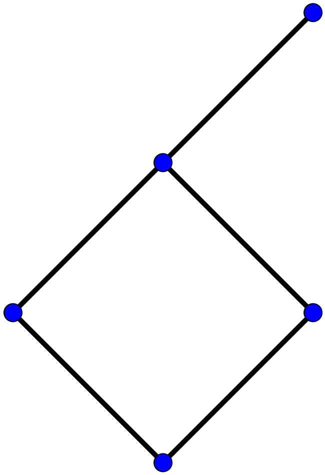
Proof.

K_5 is a non-planar graph since $e = 10 > 9 = 3n - 6$. $K_{3,3}$ is a non-planar graph since $e = 9 > 8 = 2n - 4$. □



$K_{3,3}$ K_5

Maximal planar graphs



Remark. (Maximal planar graphs / triangulations). The proof of the above theorem shows that having $3n-6$ edges in a simple n-vertex planar graph requires $2e = 3f$, meaning that every face is a triangle. If G has some face that is not a triangle, then we can add an edge between non-adjacent vertices on the boundary of this face to obtain a larger plane graph. Hence the simple plane graphs with $3n-6$ edges, the triangulations, and the maximal plane graphs are all the same family.

Thank you!