Lecture 10. Matchings

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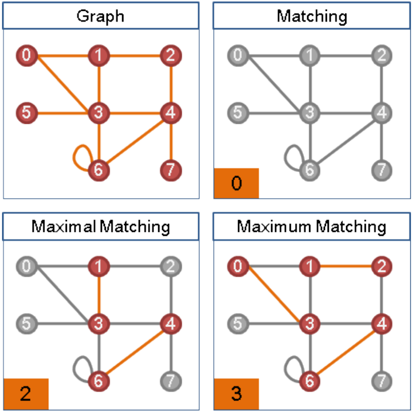
Matching

Definition

A set of edges $M \subseteq E(G)$ in a graph G is called a matching if $e \cap e' = \emptyset$ for any pair of edges $e, e' \in M$.

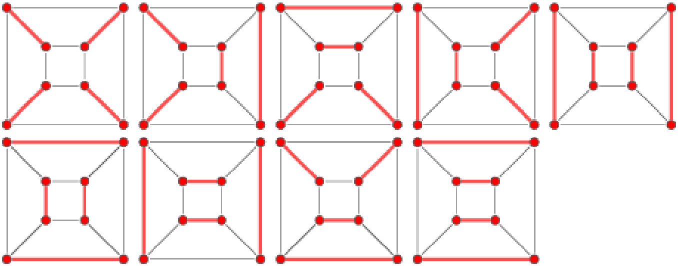


Perfect matching

Definition

A matching is perfect if $|M| = \frac{|V(G)|}{2}$, i.e. it covers all vertices of G.

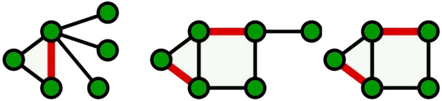
So in a perfect matching, every vertex of the graph is incident to exactly one edge of the matching. A perfect matching is therefore a matching containing $n/2$ edges (the largest possible), meaning perfect matchings are only possible on graphs with an even number of vertices.



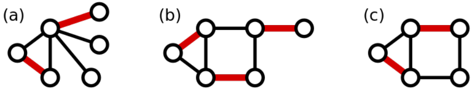
Definition

We denote the size of the maximum matching in G, by $\nu(G)$.

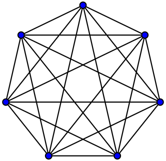
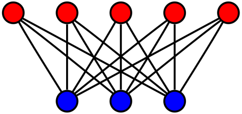
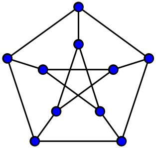
The following figure shows examples of maximal matchings (red) in three graphs.



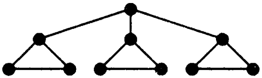
The following figure shows examples of maximum matchings (red) in the same three graphs.

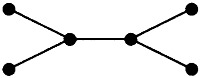


Maximum matching

- $G = K_n$  : $\nu(G) = \lfloor \frac{n}{2} \rfloor$;
- $G = K_{s,t}$, $s \leq t$  : $\nu(G) = s$;
- G is the Petersen graph  : $\nu(G) = 5$.

- A matching contains no loops;
- A matching in a graph G corresponds to an independent set in the line graph $L(G)$;
- If G has a perfect matching, then the order of G is even;
- If the order of G is even, G may NOT have a perfect matching.

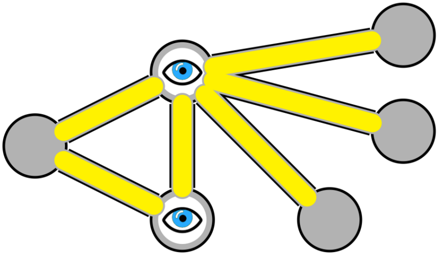




Definition

A set of vertices $T \subseteq V(G)$ of a graph G is called a cover of G if every edge $e \in E(G)$ intersects T ($e \cap T \neq \emptyset$), i.e., $G \setminus T$ is an empty graph.

A vertex cover of a graph is a set of vertices that includes at least one endpoint of every edge.

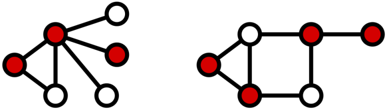


Minimum cover

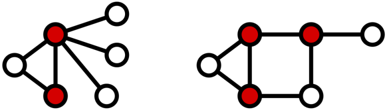
Definition

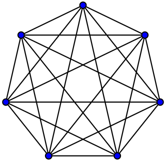
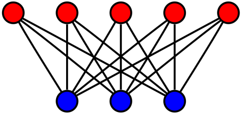
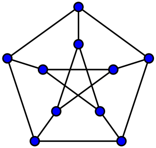
We denote the size of the minimum cover in G, by $\tau(G)$.

The following figure shows two examples of vertex covers (marked in red).



The following figure shows examples of minimum covers in the previous graphs.



- $G = K_n$  : $\tau(G) = n - 1$;
- $G = K_{s,t}$, $s \le t$  : $\tau(G) = s$;
- G is the Petersen graph  : $\tau(G) = 6$.

Note that the graphs induced by the outer 5 vertices and inner 5 vertices are both 5-cycles C_5. Since $\tau(C_5) = 3$, at least 3 of the outer vertices and 3 of the inner vertices must be included in a vertex cover.

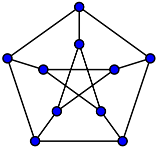
Relation between $\nu(G)$ and $\tau(G)$

Proposition

$\nu(G) \leq \tau(G) \leq 2\nu(G)$.

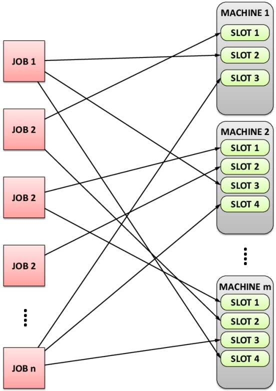
Proof.

Let M be a maximum matching in G. Since every cover has at least one vertex on each edge of M and edges are disjoint, we have $\nu(G) \leq \tau(G)$. Note also that since M is maximum, every edge $e \in E(G)$ intersects some edge $e' \in M$, otherwise we get a larger matching. So the vertices covered by M form a cover for G, hence $\tau(G) \leq 2|M| = 2\nu(G)$. □

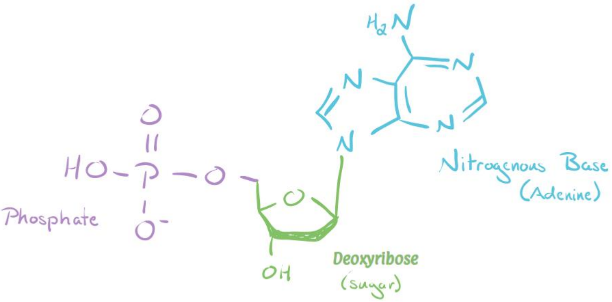


Real-world applications I

Suppose certain workers can operate certain machines, but only one at a time; this gives a bipartite graph between workers and machines. If we want to have many machines operating at the same time, we need a large matching in our bipartite graph.

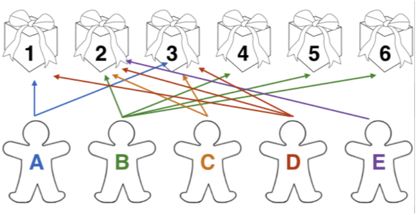


The molecular structure of a compound can be described by a graph. For certain kinds of hydrocarbon molecules, a perfect matching of this graph gives information about the location of its "double bonds".



The Stable Marriage Problem

The purpose of the stable marriage problem is to facilitate matchmaking between two sets of people. Given a list of potential matches among an equal number of brides and grooms, we hope everyone to be married to an agreeable match.

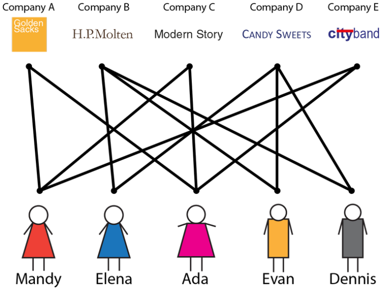


The Gale-Shapley algorithm for efficiently computing a stable matching was worth the Nobel prize in economics in 2012.

Real-world applications IV

Algorithms to find large matchings are essential subroutines for solving optimization problems. The Chinese postman problem involves travelling every edge at least once while as short a total distance as possible. This problem can be efficiently solved by first solving a set of shortest path problems, then solving a certain matching problem.

Hall's theorem



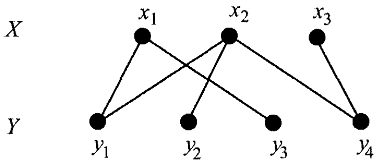
Every company can hire a suitable person if and only if: for any set of x companies, the set of applicants they choose must have a size at least x.

Hall 1935: A bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$ has a matching covering A if and only if $|N(S)| \geq |S| \quad \forall S \subseteq A$.

Hall's theorem

Let $N(S)$ be the neighbourhood of S, that is,

$$N(S) = \{v \in V(G) \mid v \text{ is adjacent to some vertex in } S\}.$$



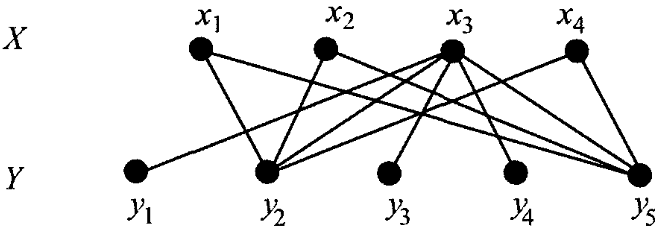
$N(\{x_1\}) = \{y_1, y_3\}$, $N(\{x_2\}) = \{y_1, y_2, y_4\}$, $N(\{x_3\}) = \{y_4\}$,

$N(\{x_1, x_2\}) = \{y_1, y_2, y_3, y_4\}$, $N(\{x_1, x_3\}) = \{y_1, y_3, y_4\}$,

$N(\{x_2, x_3\}) = \{y_1, y_2, y_4\}$,

$N(X) = Y$.

We can find a matching covering X.



Let $S = \{x_1, x_2, x_4\}$. Then $N(S) = \{y_2, y_5\}$, $|N(S)| < |S|$.

We can not find a matching covering X.

Hall's theorem

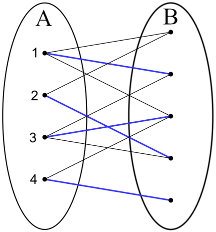
Theorem (Hall 1935)

A bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$ has a matching covering A if and only if

$$|N(S)| \geq |S| \quad \forall S \subseteq A \tag{1}$$

Proof.

It is easy to see that if G has such a matching then (1) holds.



Hall's theorem

Theorem (Hall 1935)

A bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$ has a matching covering A if and only if

$$|N(S)| \geq |S| \quad \forall S \subseteq A \tag{1}$$

Proof.

To show the other direction, we apply induction on $|A|$. For $|A| = 1$ the assertion is true. Now let $|A| \geq 2$, and assume that (1) is sufficient for the existence of a matching covering A when $|A|$ is smaller.

Proof.

If $|N(S)| \geq |S| + 1$ for every non-empty set $S \subset A$, then we pick an edge $(a, b) \in G$ and consider the graph $G' = G \setminus \{a, b\}$ obtained by deleting the vertices a and b. Then every non-empty set $S \subseteq A \setminus \{a\}$ satisfies

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|,$$

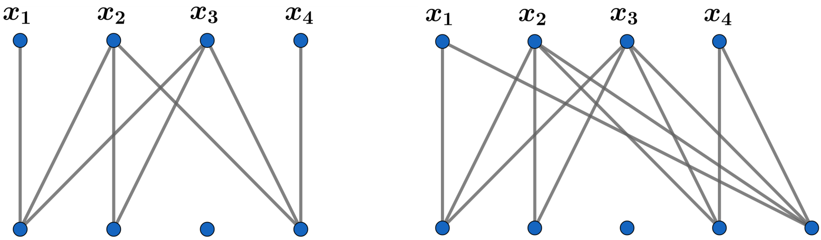
so by the induction hypothesis G' contains a matching covering $A \setminus \{a\}$. Together with the edge ab, this yields a matching covering A in G.

Proof.

Suppose now that A has a non-empty proper subset A' with neighbourhood $B' = N(A')$ such that $|A'| = |B'|$. By the induction hypothesis, $G' = G[A' \cup B']$ contains a matching covering A'. But $G \setminus G'$ satisfies (1) as well. If not, for any set $S \subseteq A \setminus A'$ with $|N_{G\setminus G'}(S)| < |S|$ we would have $|N_G(S \cup A')| = |N_{G\setminus G'}(S)| + |B'| < |S \cup A'|$, contrary to our assumption. Again, by induction, $G \setminus G'$ contains a matching of $A \setminus A'$. Putting the two matchings together, we obtain a matching in G covering A. □

Corollary

If in a bipartite graph $G = (A \cup B, E)$ we have $|N(S)| \geq |S| - d$ for every set $S \subseteq A$ and some fixed $d \in N$, then G contains a matching of cardinality $|A| - d$.



Corollary

If in a bipartite graph $G = (A \cup B, E)$ we have $|N(S)| \geq |S| - d$ for every set $S \subseteq A$ and some fixed $d \in N$, then G contains a matching of cardinality $|A| - d$.

Proof.

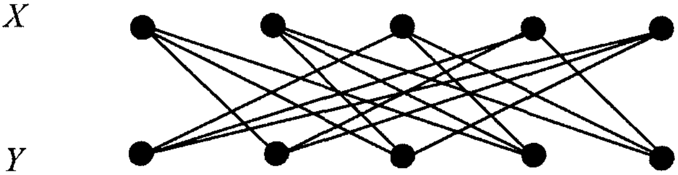
We add d new vertices to B, joining each of them to all the vertices in A. Call the resulting graph G'. Note that the new graph has

$$|N_{G'}(S)| \geq |N_G(S)| + d \geq |S| - d + d = |S|,$$

for any $S \subseteq A$, so by Hall's theorem, G' contains a matching of A. At least $|A| - d$ edges in this matching must be edges of G. □

Corollary

If a bipartite graph $G = (A \cup B, E)$ *is* k*-regular with* $k \geq 1$*, then* G *has a perfect matching.*



Regular bipartite graph

Corollary

If a bipartite graph $G = (A \cup B, E)$ is k-regular with $k \geq 1$, then G has a perfect matching.

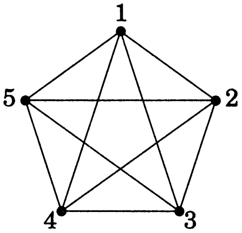
Proof.

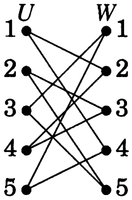
If G is k-regular, then clearly $|A| = |B|$, since the total number of edges is $k|A| = \sum_{x \in A} d(x) = \sum_{y \in B} d(y) = k|B|$. It thus suffices to show by Hall's Theorem that G contains a matching covering A. Now every set $S \subseteq A$ is joined to $N(S)$ by a total of $k|S|$ edges, and these are among the $k|N(S)|$ edges of G incident with $N(S)$. Therefore $k|S| \leq k|N(S)|$, so G does indeed satisfy (1). □

Corollary

Every regular graph of positive even degree has a 2-factor (a spanning 2-regular subgraph).

Consider the Eulerian tour in $G = K_5$ that successively visits 12314254351. The corresponding bipartite graph H is on the right.





Regular even graph

Corollary

Every regular graph of positive even degree has a 2-factor (a spanning 2-regular subgraph).

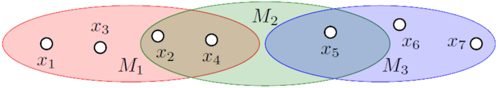
Proof.

Let G be any connected $2k$-regular graph. Then G contains an Euler tour. Define a new graph G' by splitting every vertex v into two vertices v^- and v^+. If an edge of the Euler tour goes from v to w, put an edge in G' from v^+ to w^-. So, the edges in G and in G' naturally correspond to each other. It is easy to see that G' is bipartite and k-regular so contains a perfect matching. Collapsing each pair of vertices v^-, v^+ back into a single vertex v, a perfect matching of G' corresponds to a 2-factor of G. (Each vertex v is incident to one edge which was incident to v^+ in G', and one edge incident to v^- in G'). □

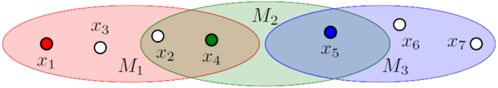
Remark. A 2-factor is a disjoint union of cycles covering all the vertices of a graph.

Definition

Let $A_1, \ldots, A_n$ be a collection of sets. A family $\{a_1, \ldots, a_n\}$ is called a system of distinct representatives (SDR) if all the a_i are distinct, and $a_i \in A_i$ for all i.



Solution.



System of distinct representatives

Corollary

A collection $A_1,\ldots,A_n$ has an SDR if and only if for all $I \subseteq [n]$ we have $|\bigcup_{i\in I} A_i| \geq |I|$.

Let $A_1 = \{x_1,x_2,x_6\}$, $A_2 = \{x_2,x_4\}$, $A_3 = \{x_1,x_6\}$, $A_4 = \{x_2,x_5\}$, $A_5 = \{x_4,x_7\}$. To find an SDR of this collection, we need to construct a bipartite graph.

Corollary

A collection $A_1,\dots,A_n$ has an SDR if and only if for all $I \subseteq [n]$ we have $|\bigcup_{i\in I} A_i| \geq |I|$.

Proof.

Define a bipartite graph with parts $A=[n]$ and $X=\bigcup_i A_i$ such that (i,a) is an edge if and only if $a \in A_i$. A matching of $[n]$ in this graph corresponds exactly to an SDR, where an edge (i,a) in the matching means that $a_i = a$. But the condition $|\bigcup_{i\in I} A_i| \geq |I|$ is precisely Hall's condition for the existence of a matching covering A, so Hall's theorem provides the desired equivalence. □

Thank you!