Lecture 9. Eulerian and Hamiltonian cycles

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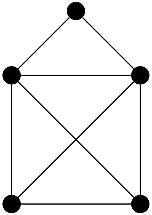
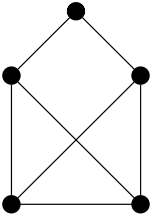
Question

Which of the two pictures below can be drawn in one go without lifting your pen from the paper?



or



Eulerian trail and Eulerian tour

Recall that a walk in G is a sequence of vertices v_0, v_1, v_2, ..., v_k, and a sequence of edges $(v_i, v_{i+1}) \in E(G)$. A path is a walk with no repeated vertices.

Definition

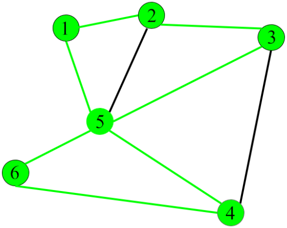
A trail is a walk with no repeated edges.

A path must be a trail; a trail must be a walk.

Definition

An Eulerian trail in a (multi)graph $G = (V, E)$ is a walk in G passing through every edge exactly once. If this walk is closed (starts and ends at the same vertex) it is called an Eulerian tour.

Eulerian trail and Eulerian tour



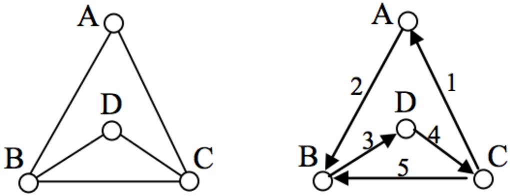
A walk: 123523456 Both vertices and edges can be repeated.

A trail: 1253456 Edges can not be repeated.

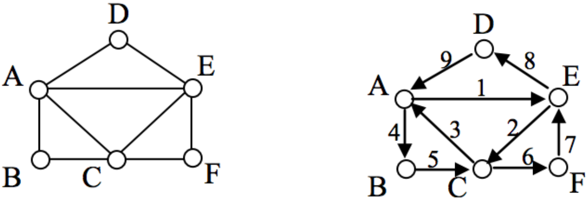
A path: 125346 Neither vertices nor edges can be repeated.

This graph has no Eulerian trail and no Eulerian tour.

Eulerian trail and Eulerian tour



We can find an Eulerian trail: *CABDCB*.

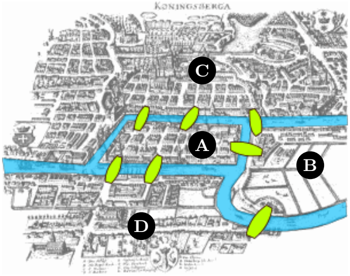


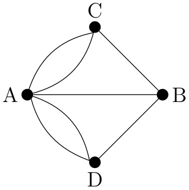
We can find an Eulerian tour: *AECABCFEDA*, which is also an Eulerian trail.

Seven bridges of Königsberg

Question

Is it possible to design a closed walk passing through all the 7 bridges exactly once? Equivalently, does the graph below have an Eulerian walk?





Necessary and sufficient condition of an Eulerian tour

Theorem

A connected (multi)graph has an Eulerian tour if and only if each vertex has even degree.

In order to prove this theorem we use the following lemma.

Lemma

Every maximal trail in an even graph (i.e., a graph where all the vertices have even degree) is a closed trail.

Proof.

Let T be a maximal trail. If T is not closed, then T has an odd number of edges incident to the final vertex v. However, as v has even degree, there is an edge incident to v that is not in T. This edge can be used to extend T to a longer trail, contradicting the maximality of T. □

Necessary and sufficient condition of an Eulerian tour

Theorem

A connected (multi)graph has an Eulerian tour if and only if each vertex has even degree.

Proof.

To see that the condition is necessary, suppose G has an Eulerian tour C. If a vertex v was visited k times in the tour C, then each visit used 2 edges incident to v (one incoming edge and one outgoing edge). Thus, $d(v) = 2k$, which is even.

Necessary and sufficient condition of an Eulerian tour

Theorem

A connected (multi)graph has an Eulerian tour if and only if each vertex has even degree.

Proof.

To see that the condition is sufficient, let G be a connected graph with even degrees. Let $T = e_1 e_2 \dots e_\ell$ (where $e_i = (v_{i-1}, v_i)$) be a longest trail in G. Then, by the last lemma, T is closed, i.e., $v_0 = v_\ell$. If T does not include all the edges of G then, since G is connected, there is an edge e outside of T such that $e = (u, v_i)$ for some vertex v_i in T. But then $T' = e e_{i+1} \dots e_\ell e_1 e_2 \dots e_i$ is a trail in G which is longer than T, contradicting the fact that T is a longest trail in G. Thus, we conclude that T includes all the edges of G and so it is an Eulerian tour. □

Corollary

A connected multigraph G has an Eulerian trail if and only if it has either 0 or 2 vertices of odd degree.

Proof.

Suppose T is an Eulerian trail from vertex u to vertex v. If $u = v$ then T is an Eulerian tour and so it follows from the theorem that all the vertices in G have even degree. If $u \neq v$, note that the multigraph $G \cup \{e\}$, where $e = (u, v)$ is a new edge, has an Eulerian tour, namely $T \cup \{e\}$. It follows from the theorem that all the degrees in $G \cup \{e\}$ are even. Thus, we conclude that, in the original multigraph G, the vertices u, v are the only ones which have odd degree.

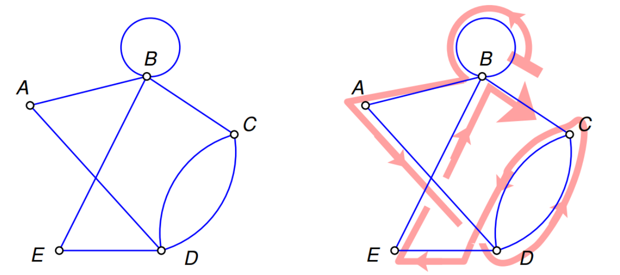
Corollary

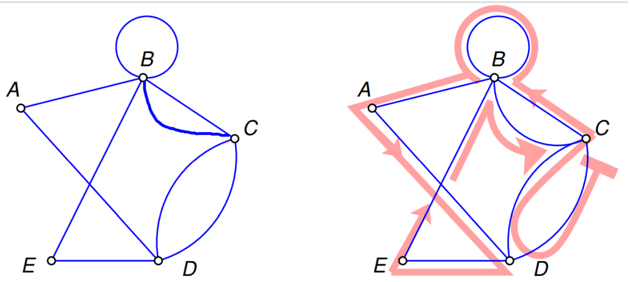
A connected multigraph G has an Eulerian trail if and only if it has either 0 or 2 vertices of odd degree.

Proof.

Now we prove the other direction of the corollary. If G has no vertices of odd degree then from the theorem it contains an Eulerian tour which is also an Eulerian trail. Suppose now that G has 2 vertices u,v of odd degree. Then $G \cup \{e\}$, where $e = (u,v)$ is a new edge, only has vertices of even degree and so, by the last theorem, it has an Eulerian tour C. Removing the edge e from C gives an Eulerian trail of G from u to v. □

Eulerian trail and Eulerian tour in multigraphs





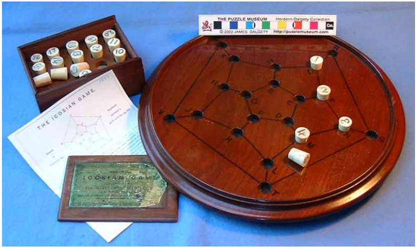
Hamilton paths and cycles

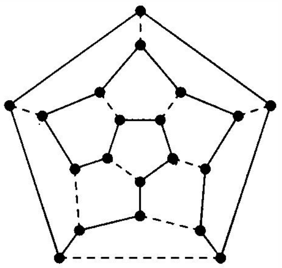
Definition

A Hamilton path/cycle in a graph G is a path/cycle visiting every vertex of G exactly once. A graph G is called Hamiltonian if it contains a Hamilton cycle.

Hamilton cycles were introduced by Kirkman in 1856, and were named after Sir William Hamilton, who produced a puzzle whose goal was to find a Hamilton cycle in a specific graph.

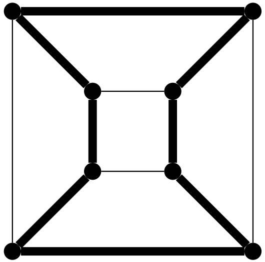
The Icosian Game





Example

Hamilton cycle in the skeleton of the 3-dimensional cube.

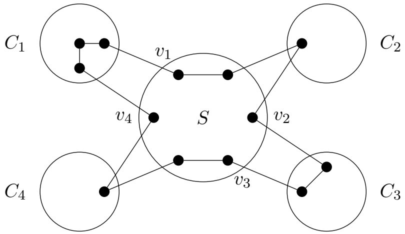


Necessary conditions for Hamiltonicity

Every Hamiltonian graph is 2-connected.

Proposition

If G is Hamiltonian then for any set $S \subseteq V$ the graph $G \setminus S$ has at most $|S|$ connected components.



Necessary conditions for Hamiltonicity

Proposition

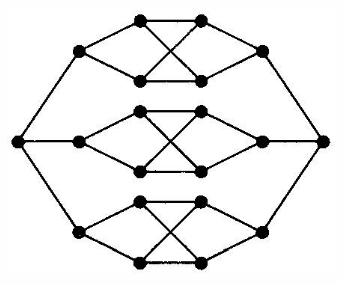
If G is Hamiltonian then for any set $S \subseteq V$ the graph $G \setminus S$ has at most $|S|$ connected components.

Proof.

Let $C_1, \ldots, C_k$ be the components of $G \setminus S$. Imagine that we are moving along a Hamilton cycle in some order, vertex-by-vertex (we are moving clockwise, starting from some vertex in C_1, say). We must visit each component of $G \setminus S$ at least once; when we leave C_i for the first time, let v_i be the subsequent vertex visited (which must be in S). Each v_i must be distinct because a cycle cannot intersect itself. Hence, S must have at least as many vertices as the number of connected components of $G \setminus S$. □

Proposition

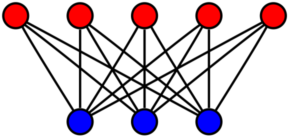
If G is Hamiltonian then for any set $S \subseteq V$ the graph $G \setminus S$ has at most $|S|$ connected components.



From the Proposition we see that this graph is not Hamiltonian.

Corollary

If a connected bipartite graph $G = (V, E)$ *with bipartition* $V = A \cup B$ *is Hamiltonian then* $|A| = |B|$.



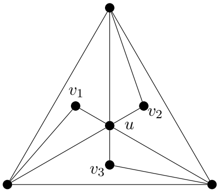
Proof.

By deleting the vertices in A from G we get $|B|$ isolated vertices and so $G \setminus A$ has $|B|$ connected components. Thus, by Proposition 4.10 we conclude that $|A| \geq |B|$. By symmetry we can also show that $|B| \geq |A|$. Thus, we conclude that $|A| = |B|$. □

Proposition 4.10 is not sufficient

Example

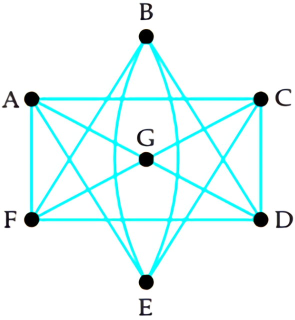
The condition in Proposition 4.10 is not sufficient to ensure that a graph is Hamiltonian. The graph G below satisfies the condition of Proposition 4.10 but is not Hamiltonian. Indeed, one would need to include all the edges incident to the vertices v_1, v_2 and v_3 in a Hamilton cycle of G; however, in that case the vertex u would have degree at least 3 in that Hamilton cycle, which is impossible.



Dirac's theorem

Theorem (Dirac 1952)

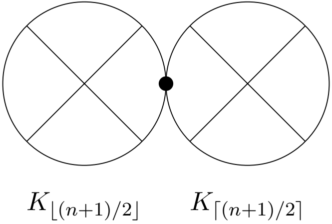
If G is a simple graph with $n \geq 3$ vertices and if $\delta(G) \geq n/2$, then G is Hamiltonian.



Best-possible minimum degree bound

Example

The graph consisting of two cliques of orders $\lfloor (n+1)/2 \rfloor$ and $\lceil (n+1)/2 \rceil$ sharing a vertex has minimum degree $\lfloor (n-1)/2 \rfloor$ but is not Hamiltonian (it is not even 2-connected).



Example

If n is odd, then the complete bipartite graph $K_{(n-1)/2,(n+1)/2}$ has minimum degree $(n-1)/2$ but is not Hamiltonian.

Proof of Dirac's theorem

Proof.

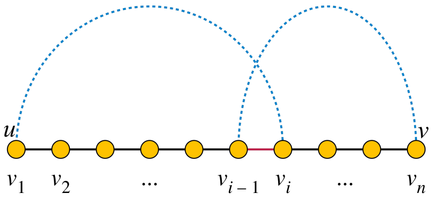
The condition that $n \geq 3$ must be included since K_2 is not Hamiltonian but satisfies $\delta(K_2) = |K_2|/2$.

If there is a non-Hamiltonian graph satisfying the hypotheses, then adding edges cannot reduce the minimum degree, so we may restrict our attention to *maximal* non-Hamiltonian graphs G with minimum degree at least $n/2$. By "maximal" we mean that for every pair (u, v) of non-adjacent vertices of G, the graph obtained from G by adding the edge $e = (u, v)$ is Hamiltonian.

Proof of Dirac's theorem

Proof.

The maximality of G implies that G has a Hamilton path, say from $u = v_1$ to $v = v_n$, because every Hamilton cycle in $G \cup e$ must contain the new edge e. We use most of this path $v_1, \ldots, v_n$, with a small switch, to obtain a Hamilton cycle in G. If some neighbour of u immediately follows a neighbour of v on the path, say $(u, v_i) \in E(G)$ and $(v, v_{i-1}) \in E(G)$, then G has the Hamilton cycle $(u, v_i, v_{i+1}, \ldots, v_{n-1}, v, v_{i-1}, v_{i-2}, \ldots, v_2, u)$ shown below.



Proof of Dirac's theorem

Proof.

To prove that such a cycle exists, we show that there is a common index in the sets S and T defined by $S = \{i : (u, v_{i+1}) \in E(G)\}$ and $T = \{i : (v, v_i) \in E(G)\}$. Summing the sizes of these sets yields

$$|S \cup T| + |S \cap T| = |S| + |T| = d(u) + d(v) \geq n$$

Neither S nor T contains the index n. This implies that $|S \cup T| < n$, and hence $|S \cap T| \geq 1$, as required. This is a contradiction. □

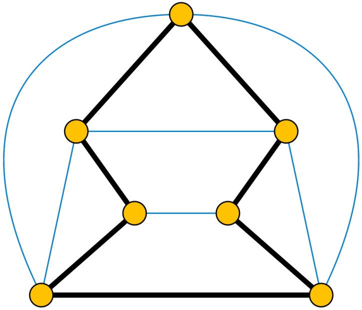
Ore observed that this argument uses only that $d(u)+d(v) \geq n$. Therefore, we can weaken the requirement of minimum degree $n/2$ to require only that $d(u)+d(v) \geq n$ whenever u is not adjacent to v.

Theorem (Ore 1960)

If G is a simple graph with $n \geq 3$ vertices such that for every pair of non-adjacent vertices u, v of G we have $d(u)+d(v) \geq n$, then G is Hamiltonian.

Ore's theorem

A graph meeting the conditions of Ore's theorem: for every pair of non-adjacent vertices u, v of G we have $d(u) + d(v) \geq n$.



Thank you!