Lecture 8. Menger's theorem

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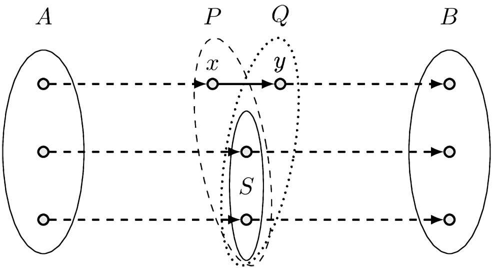
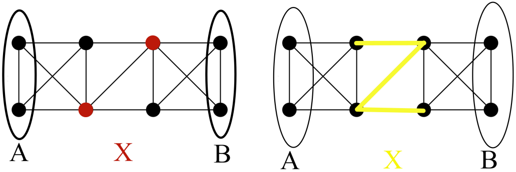
Definition

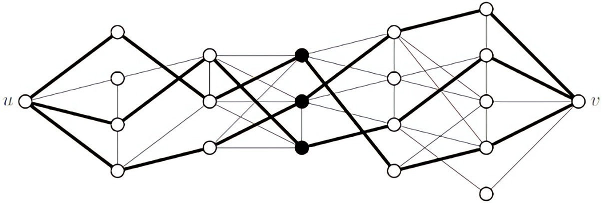
Let $A, B \subseteq V$. An $A-B$ path is a path with one endpoint in A, the other endpoint in B, and all interior vertices outside of $A \cup B$. Any vertex in $A \cap B$ is a trivial $A-B$ path.



$A - B$ path

If $X \subseteq V$ (or $X \subseteq E$) is such that every $A - B$ path in G contains a vertex (or an edge) from X, we say that X separates the sets A and B in G. This implies in particular that $A \cap B \subseteq X$.





Theorem (Menger's theorem)

Let $G = (V, E)$ be a graph and let $S, T \subseteq V$. Then the maximum number of vertex-disjoint $S - T$ paths is equal to the minimum size of an $S - T$ separating vertex set.

Proof.

Obviously, the maximum number of disjoint paths does not exceed the minimum size of a separating set, because for any collection of disjoint paths, any separating set must contain a vertex from each path. So we just need to prove there is an $S - T$ separating set and a collection of disjoint $S - T$ paths with the same size.

Because the maximum number of vertex-disjoint $S-T$ paths

$\leq$ the minimum size of an $S-T$ separating vertex set

To prove the maximum number of vertex-disjoint $S-T$ paths

$=$ the minimum size of an $S-T$ separating vertex set

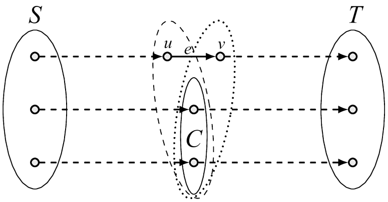
We need an $S-T$ separating set and a collection of disjoint $S-T$ paths with the same size.

Proof.

We use induction on $|E|$, the case $E=\emptyset$ being trivial. We first consider the case where S and T are disjoint.

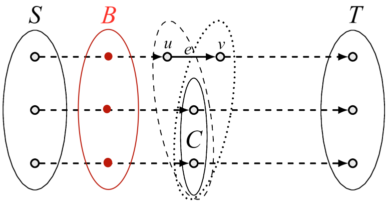
Let k be the minimum size of an $S-T$ separating vertex set.

Choose $e=(u,v)\in E$. Let $G'=(V,E\setminus e)$. If each $S-T$ separating vertex set in G' has size at least k, then inductively there exist k vertex-disjoint $S-T$ paths in G', hence in G.

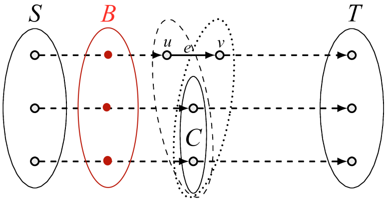


So we can assume that G' has an $S-T$ separating vertex set C of size at most $k-1$. Then $C \cup \{u\}$ and $C \cup \{v\}$ are $S-T$ separating vertex sets of G of size k.

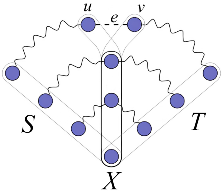
Since C is a separating set for G', no component of $G' \setminus C$ has elements from both S and T. Let V_S be the union of components with elements from S, and let V_T be the union of components with elements in T. If we were to add the edge (u,v) to $G' \setminus C$ then there would be a path from S to T (because C does not separate S and T in G). So, without loss of generality $u \in V_S$ and $v \in V_T$.



Now, each $S-(C\cup\{u\})$ separating vertex set B of G' has size at least k, as it is $S-T$ separating in G. Indeed, each $S-T$ path P in G intersects $C\cup\{u\}$. Let P' be the subpath of P that goes from S to the first time it touches $C\cup\{u\}$. If P' ends with a vertex in C, then $u\notin P'$ so P' is an $S-(C\cup\{u\})$ path in G'. If P' ends in u, then it is disjoint from C and so by the above it contains only vertices in V_S. So $v\notin P'$ and again P' is an $S-(C\cup\{u\})$ path in G'. In both cases we showed that P' is an $S-(C\cup\{u\})$ path in G' so P intersects B.



So by induction, G' contains k disjoint $S-(C\cup\{u\})$ paths. Similarly, G' contains k disjoint $(C\cup\{v\})-T$ paths. Any path in the first collection intersects any path in the second collection only in C, since otherwise G' contains an $S-T$ path avoiding C. Hence, as $|C|=k-1$, we can pairwise concatenate these paths to obtain $k-1$ disjoint $S-T$ paths. We can finally obtain a kth path by inserting the e between the path ending at u and the path starting at v.

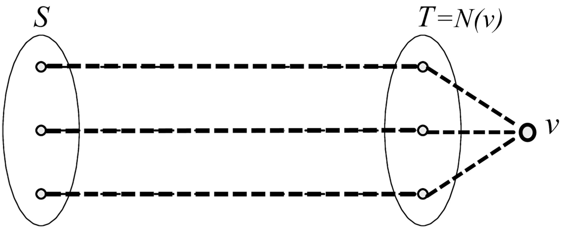


It remains to consider the general situation where S and T might not be disjoint. Let $X = S \cap T$ and apply the theorem with the disjoint sets $S' = S \setminus X$ and $T' = T \setminus X$, in the graph $G' = G \setminus X$. Let k' be the size of a minimum separating set in G'. We can obtain a $k' + |X|$-vertex $S - T$ separating set in G by adding every vertex in X to an $S' - T'$ separating set in G'. Similarly we can obtain a collection of $k' + |X|$ vertex-disjoint $S - T$ paths by adding each vertex in X as a trivial path to a collection of vertex-disjoint $S' - T'$ paths in G'.

Corollary

Corollary

For $S \subseteq V$ and $v \in V \setminus S$, the minimum number of vertices distinct from v separating v from S in G is equal to the maximum number of paths forming a $v-S$ fan in G. (that is, the maximum number of $\{v\}-S$ paths which are disjoint except at v).



Corollary

For $S \subseteq V$ and $v \in V \setminus S$, the minimum number of vertices distinct from v separating v from S in G is equal to the maximum number of paths forming a $v - S$ fan in G. (that is, the maximum number of $\{v\} - S$ paths which are disjoint except at v).

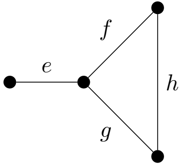
Proof.

Apply Menger's Theorem with $T = N(v)$. Note that none of the resulting paths go through v; if one did, then it would contain two vertices of T, violating the definition of an $S - T$ path. So we have a suitable number of vertex-disjoint $S - T$ paths not including v, and we can append v to each path to give a $v - S$ fan. □

Line graph

Definition

The line graph of G, written $L(G)$, is the graph whose vertices are the edges of G, with $(e,f) \in E(L(G))$ when $e = (u,v)$ and $f = (v,w)$ in G (i.e. when e and f share a vertex).



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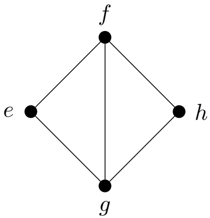

$L(G)$

Corollary

Corollary

Let u and v be two distinct vertices of G.
1. If $(u,v) \notin E$, then the minimum number of vertices different from u,v separating u from v in G is equal to the maximum number of internally vertex-disjoint $u-v$ paths in G.
2. The minimum number of edges separating u from v in G is equal to the maximum number of edge-disjoint $u-v$ paths in G.

Proof.

For (1), apply Menger's Theorem with $S = N(u)$ and $T = N(v)$.
For (2), apply Menger's Theorem to the line graph of G, with S as the set of edges adjacent to u and T as the set of edges adjacent to v. □

Global version of Menger's theorem

Theorem (Global version of Menger's theorem)

1. A graph is k-connected if and only if it contains k internally vertex-disjoint paths between any two vertices.
2. A graph is k-edge-connected if and only if it contains k edge-disjoint paths between any two vertices.

Proof.

We need only to prove (1). Then (2) follows straight from the above corollary.
For (1), if a graph G contains k internally disjoint paths between any two vertices, then $|G| > k$ and G cannot be separated by fewer than k vertices; thus, G is k-connected.

Proof.

Conversely, suppose that G is k-connected (and, in particular, has more than k vertices) but contains vertices u,v not linked by k internally disjoint paths. By the above corollary, u and v are adjacent; let $G' = G \setminus (u,v)$. Then G' contains at most $k-2$ internally disjoint u,v-paths. By the above corollary, we can separate u and v in G' by a set X of at most $k-2$ vertices. As $|G| > k$, there is at least one further vertex $w \notin X \cup \{u,v\}$ in G. Now X separates w in G' from either u or v (say, from u). But then $X \cup \{v\}$ is a set of at most $k-1$ vertices separating w from u in G, contradicting the k-connectedness of G. □

Thank you!