Lecture 6. Vertex connectivity

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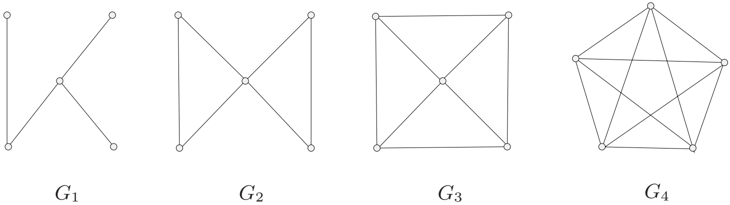
Definition

A vertex cut in a connected graph $G = (V, E)$ is a set $S \subseteq V$ such that $G \setminus S := G[V \setminus S]$ has more than one connected component. A cut vertex is a vertex v such that $\{v\}$ is a cut.



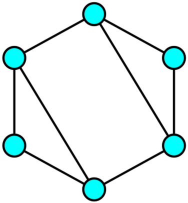
k-connected and connectivity

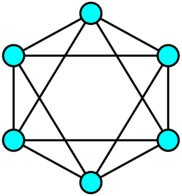
Definition

G is called k-connected if $|V(G)| > k$ and if $G \setminus X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices of G are separated by fewer than k other vertices. Every (non-empty) graph is 0-connected. The 1-connected graphs are precisely the non-trivial connected graphs. The greatest integer k such that G is k-connected is the connectivity $\kappa(G)$ of G.

Remark. K_1 is connected but not 1-connected.
Except for K_1, 'connected graph'='1-connected graph'.
In some other literatures, K_1 is also 1-connected.

k-connected and connectivity

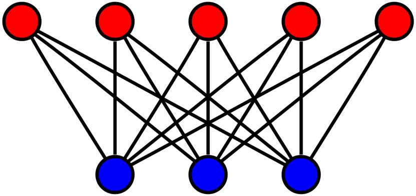




The left graph is 1-connected and 2-connected. Its connectivity $\kappa(G) = 2$.

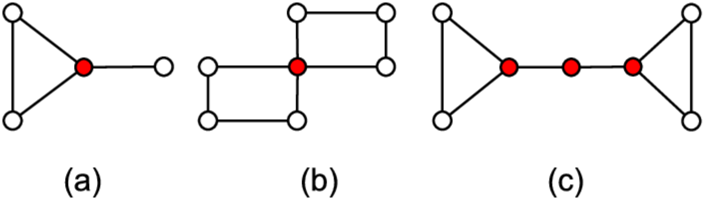
The right graph is 1-connected, 2-connected, 3-connected and 4-connected. Its connectivity $\kappa(G) = 4$.

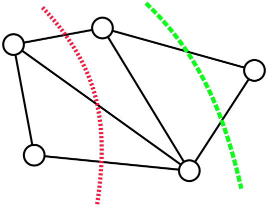
$G = K_n$: $\kappa(G) = n - 1$.



$G = K_{m,n}$, $m \leq n$: $\kappa(G) = m$. Indeed, let G have bipartition $A \cup B$, with $|A| = m$ and $|B| = n$. Deleting A disconnects the graph. On the other hand, deleting $S \subseteq V$ with $|S| < m$ leaves both $A \setminus S$ and $B \setminus S$ non-empty and any $A \setminus S$ is connected to any $B \setminus S$. Hence, $G \setminus S$ is connected.

Cut vertex and vertex cut





Connectivity and minimum degree

Proposition

For every graph G, $\kappa(G) \leq \delta(G)$.

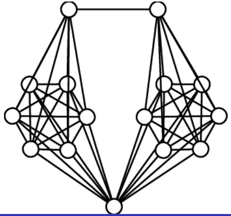
Proof.

If G is a complete graph then trivially $\kappa(G) = \delta(G) = |G| - 1$.

Otherwise let $v \in G$ be a vertex of minimum degree $d(v) = \delta(G)$.

Deleting $N(v)$ disconnects v from the rest of G. □

Remark. High minimum degree does not imply high connectivity. Consider two disjoint copies of K_n.



Theorem (Mader's theorem)

Every graph of average degree at least $4k$ has a k-connected subgraph.

Proof.

For $k \in \{0,1\}$ the assertion is trivial; we consider $k \geq 2$ and a graph $G = (V,E)$ with $|V| = n$ and $|E| = m$. For inductive reasons it will be easier to prove the stronger assertion that G has a k-connected subgraph whenever

- (i) $n \geq 2k-1$ and
- (ii) $m \geq (2k-3)(n-k+1)+1$.

(This assertion is indeed stronger, i.e. (i) and (ii) follow from our assumption of $\overline{d}(G) \geq 4k$: (i) holds since $n > \Delta(G) \geq 4k$, while (ii) follows from $m = \frac{1}{2}\overline{d}(G)n \geq 2kn$.)

Theorem (Stronger assertion)

Let $G = (V, E)$ with $|V| = n$ and $|E| = m$. If $n \geq 2k - 1$, $m \geq (2k-3)(n-k+1)+1$, then G has a k-connected subgraph.

Proof.

We apply induction on n. If $n = 2k - 1$, then $k = \frac{1}{2}(n+1)$, and hence

$$m \geq (n-2)\frac{n+1}{2} + 1 = \frac{1}{2}n(n-1)$$

by (ii). Thus $G = K_n \supseteq K_{k+1}$, proving our claim.

Proof.

We therefore assume that $n \geq 2k$.

If v is a vertex with $d(v) \leq 2k-3$, then $G \setminus v$ has $n-1$ vertices and at least $(2k-3)[(n-1)-k+1]+1$ edges. We can apply the induction hypothesis to $G \setminus v$ and are done. So we assume that $\delta(G) \geq 2k-2$.

If G is itself not k-connected, then there is a separating set $X \subseteq V$ with less than k vertices, such that $G \setminus X$ has two components on the vertex sets V_1, V_2. Let $G_i = G[V_i \cup X]$, so that $G = G_1 \cup G_2$, and every edge of G is either in G_1 or G_2 (or both). Each vertex in each V_i has at least $\delta(G) \geq 2k-2$ neighbours in G and thus also in G_i, so $|G_1|, |G_2| \geq 2k-1$. Note that each $|G_i| < n$, so by the induction hypothesis, if no G_i has a k-connected subgraph then each

$$e(G_i) \leq (2k-3)(|G_i|-k+1)$$

Stronger assertion

Proof.

Hence,

$$\begin{aligned} m &\le e(G_1) + e(G_2) \\ &\le (2k-3)(|G_1| + |G_2| - 2k + 2) \\ &\le (2k-3)(n-k+1) \qquad (\text{since } |G_1 \cap G_2| \le k-1), \end{aligned}$$

contradicting (ii). □

Thank you!