Lecture 5. Cayley's formula: A second proof

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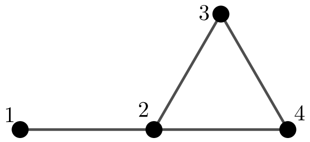
Directed graph

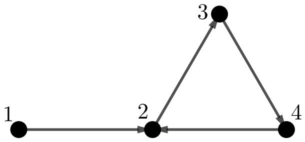
Recall that a graph G is a pair $G = (V,E)$ where V is a set of vertices and E is a (multi)set of unordered pairs of vertices.

Definition

A directed graph G, or digraph for short, is a pair $G = (V,E)$ where V is a set of vertices and E is a (multi)set of ordered pairs of vertices. Equivalently, a digraph is a (possibly not-simple) graph where each edge is assigned a direction.



A graph

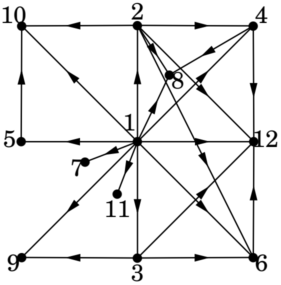


A digraph

Directed graph

Definition

Let v be a vertex in a digraph. The outdegree $d^+(v)$ is the number of edges with tail v. The indegree $d^-(v)$ is the number of edges with head v.



In the above digraph, $d^+(10) = 0, d^-(10) = 3, d^+(4) = 2, d^+(4) = 2.$

Sketch of the first proof

Theorem (Cayley's formula)

There are n^{n-2} trees with vertex set $[n]$.

Sketch of the first proof.

Iteratively delete the leaf with the smallest label and append the label of its neighbour to the sequence. After $n-2$ iterations a single edge remains and we have produced a Prüfer sequence $f(T)$ of length $n-2$.

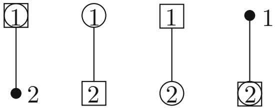
We prove that there is a bijection from the set of all trees on n vertices onto their Prüfer sequences. Because there are n^{n-2} Prüfer sequences, our proof is done. □

Theorem (Cayley's formula)

There are n^{n-2} trees with vertex set $[n]$.

Second proof, Joyal 1981.

Let t_n be the number of labelled trees on n vertices. We need to prove that $t_n = n^{n-2}$. For each labelled tree, we choose two vertices from the tree, called L and R (L and R can be the same vertex). Let T_n be the family of labelled trees with two distinguished vertices L and R. Clearly, $|T_n| = t_n n^2$, and it is thus enough to prove that $|T_n| = n^n$.



The four trees of $\mathcal{T}_2$

Second proof (continued)

Second proof.

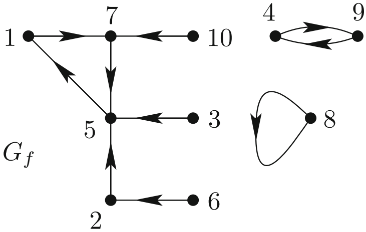
(continued) We know that the number of all mappings $f:[n]\to[n]$ is n^n. To prove $|T_n| = n^n$, we'll describe a bijection between the set of all mappings $f:[n]\to[n]$, and T_n.

So, let $f:[n]\to[n]$ be a mapping. We represent f as a directed graph G_f with vertex set $[n]$ and the set of directed edges $E(G_f) = \{(i, f(i)) \mid 1 \le i \le n\}$.

$$f = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}, \text{ where } 1 \le a_i \le n.$$

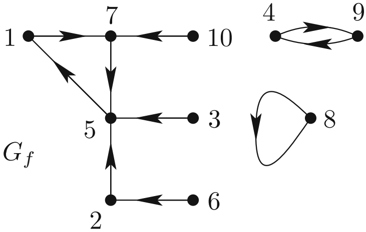
Second proof (continued)

Example. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$



(continued) G_f is a digraph in which the outdegree of every vertex is exactly one. In each component of G_f, the number of vertices equals the number of edges, and hence each component contains precisely one directed cycle.

Example. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$



(continued) Let M be the union of the vertex sets of these cycles. In order to create a tree, we need to get rid of these cycles. It is easy to see that f restricted to M is a bijection; moreover, M is the unique maximal set on which f acts as a bijection.

Second proof (continued)

Example. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$

$$M = (1,4,5,7,8,9)$$

$$f|_M = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}$$

Write $f|_M$ such that the numbers in the first row appear in natural order ($1 < 4 < 5 < 7 < 8 < 9$). This gives us an ordering of M according to the second row. For $f|_M = \begin{pmatrix} v_1 & \dots & v_k \\ f(v_1) & \dots & f(v_k) \end{pmatrix}$ such that $v_1 < v_2 < \cdots < v_k$, we can choose $f(v_1)$ as the vertex L, $f(v_k)$ as the vertex R. The tree T corresponding to f is constructed as follows: Draw a path $f(v_1), f(v_2), \dots, f(v_k)$, and fill in the remaining vertices as in G_f.

Example. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$

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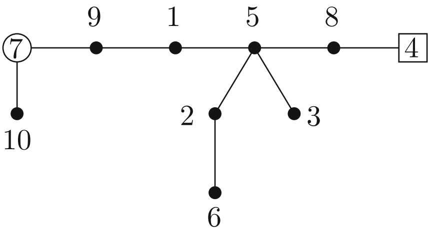
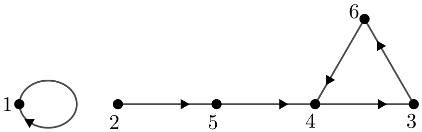


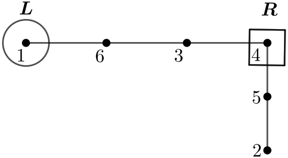
It is immediate how to reverse this correspondence: Given a tree T with two distinguished vertices L and R, we look at the unique path P from the left end to the right end. This gives us the set M and the mapping $f|_M$. The remaining correspondences $i \to f(i)$ are then filled in according to the unique paths from i to P. $\square$

Get a tree with two distinguished vertices from a mapping

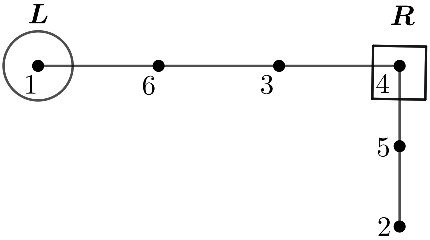
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 3 & 4 & 4 \end{pmatrix}$$



$$M = \{1, 3, 4, 6\}, \quad f|_M = \begin{pmatrix} 1 & 3 & 4 & 6 \\ 1 & 6 & 3 & 4 \end{pmatrix}$$



Get a mapping from a tree with two distinguished vertices



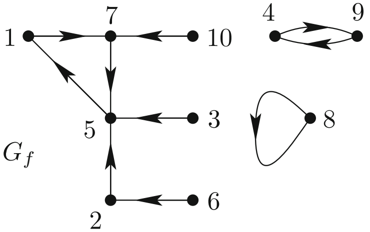
$$f|_M = \begin{pmatrix} 1 & 3 & 4 & 6 \\ 1 & 6 & 3 & 4 \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 3 & 4 & 4 \end{pmatrix}$$

Thank you!

Second proof (continued)

Example. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$



(continued) G_f is a digraph in which the outdegree of every vertex is exactly one. In each component of G_f, the number of vertices equals the number of edges, and hence each component contains precisely one directed cycle.