Lecture 5. Cayley's formula: A second proof

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Directed graph

Recall that a graph *G* is a pair $G = (V, E)$ where *V* is a set of vertices and *E* is a (multi)set of unordered pairs of vertices.

Definition

A directed graph *G*, or digraph for short, is a pair $G = (V, E)$ where V is a set of vertices and E is a (multi)set of ordered pairs of vertices. Equivalently, a digraph is a (possibly not-simple) graph where each edge is assigned a direction.

also preserve 2-edge-connectedness.

entering 12 accomplishes this.

Definition **4.2.3.** *An example of digraph connectivity.* In the digraph *G* with vertex

Let v be a vertex in a digraph. The outdegree $d^+(v)$ is the number of edges with tail *v*. The indegree $d^-(v)$ is the number of edges with head v . \blacksquare parameters from 1 to 12 through 2,3,4,6 and directly, it is necessary to delete at $\frac{1}{2}$

In the above digraph, $d^+(10) = 0, d^-(10) = 3, d^+(4) = 2, d^+(4) = 2$.

Theorem (Cayley's formula)

There are nn−² *trees with vertex set* [*n*]*.*

Sketch of the first proof.

Iteratively delete the leaf with the smallest label and append the label of its neighbour to the sequence. After *n*−2 iterations a single edge remains and we have produced a Prüfer sequence *f*(*T*) of length $n-2$. We prove that there is a bijection from the set of all trees on *n*

vertices onto their Prüfer sequences. Because there are *n n*−2

Prüfer sequences, our proof is done.

Second proof

Theorem (Cayley's formula)

There are nn−² *trees with vertex set* [*n*]*.*

Second proof, Joyal 1981.

Let t_n be the number of labelled trees on *n* vertices. We need to prove that $t_n = n^{n-2}$. For each labelled tree, we choose two vertices from the tree, called L and R (L and R can be the same vertex). Let T_n be the family of labelled trees with two distinguished vertices L and R . Clearly, $|T_n| = t_n n^2$, and it is thus enough to prove that $|T_n| = n^n$. First proof (Bijection). The classical and most direct method is to find \overline{a} bijection from the set of all trees on \overline{a} (a1,...,aⁿ−2) with ¹ [≤] ^aⁱ [≤] ⁿ comes into mind. Thus we want to uniquely encode even \mathcal{L}

The four trees of \mathcal{T}_2

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was found by Prüfer and is contained in most books on graph theory.

 $T(\tfrac{1}{2}) + T(\tfrac{1}{2}) = \tfrac{1}{2} + T(\tfrac{1}{2})$ $T(\tfrac{1}{2}) + T(\tfrac{1}{2}) = \tfrac{1}{2} + T(\tfrac{1}{2})$

Second proof.

(continued) We know that the number of all mappings $f:[n] \to [n]$ is n^n . To prove $|T_n| = n^n$, we'll describe a bijection between the set of all mappings $f:[n] \to [n]$, and T_n . So, let $f : [n] \rightarrow [n]$ be a mapping. We represent f as a directed graph G_f with vertex set [n] and the set of directed edges $E(G_f) = \{(i, f(i)) \mid 1 \le i \le n\}.$

$$
f = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}
$$
, where $1 \le a_i \le n$.

Example.
$$
f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}
$$

(continued) G_f is a digraph in which the outdegree of every vertex is exactly one. In each component of G_f , the number of vertices equals the number of edges, and hence each component [le](#page-7-0)[f](#page-12-0)[t](#page-3-0) [e](#page-4-0)[n](#page-4-0)[d](#page-11-0) [a](#page-3-0)nd f([z](#page-0-0)) [is](#page-15-0) one right end. contains precisely one directed cycle.

Our goal is thus to prove |Tn[|] ⁼ ⁿⁿ. Now there is a set whose size is known to be nⁿ, namely the set N ^N of all mappings from N into N. Thus

Example.
$$
f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}
$$

(continued) Let M be the union of the vertex sets of these cycles. In order to create a tree, we need to get rid of these cycles. It is easy to see that f restricted to M is a bijection; moreover, M is the [le](#page-8-0)[f](#page-12-0)[t](#page-3-0) [e](#page-4-0)[n](#page-4-0)[d](#page-11-0) [a](#page-3-0)nd f([z](#page-0-0)) [is](#page-15-0) one right end. unique maximal set on which *f* acts as a bijection.

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$$

\n
$$
M = (1, 4, 5, 7, 8, 9)
$$
\n
$$
f|_M = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}
$$

Write $f|_M$ such that the numbers in the first row appear in natural order $(1 < 4 < 5 < 7 < 8 < 9)$. This gives us an ordering of *M* according to the second row. For $f|_M =$ $\begin{pmatrix} v_1 & \dots & v_k \end{pmatrix}$ $f(v_1)$... $f(v_k)$! such that $v_1 < v_2 < \cdots < v_k$, we can choose $f(v_1)$ as the vertex $L, f(v_k)$ as the vertex \vec{R} . The tree \vec{T} corresponding to \vec{r} is constructed as follows: Draw a path $f(v_1), f(v_2), \ldots, f(v_k)$, and fill in the remaining vertices as in *G^f* .

Example.
$$
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In the m[a](#page-4-0)ppi[n](#page-11-0)g fille of the remaining correspon[d](#page-12-0)ence in [th](#page-0-0)e following correspondence in the following correspondence

It is immediate how to reverse this correspondence: Given a tree *T* with two distinguished vertices *L* and *R*, we look at the unique path *P* from the left end to the right end. This gives us the set *M* and the mapping $f|_M$. The remaining correspondences $i \rightarrow f(i)$ are then filled in according to the unique paths from *i* to *P*. П

Get a tree with two distinguished vertices from a mapping

$$
M = \{1, 3, 4, 6\}, f|_M = \begin{pmatrix} 1 & 3 & 4 & 6 \\ 1 & 6 & 3 & 4 \end{pmatrix}
$$

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Get a mapping from a tree with two distinguished vertices

$$
f|_M = \begin{pmatrix} 1 & 3 & 4 & 6 \\ 1 & 6 & 3 & 4 \end{pmatrix}
$$

$$
f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 3 & 4 & 4 \end{pmatrix}
$$

Thank you!

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Example.
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