Lecture 4. Cayley's formula and Prüfer code

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Question

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What is the number of spanning trees in a labelled complete graph on n vertices?

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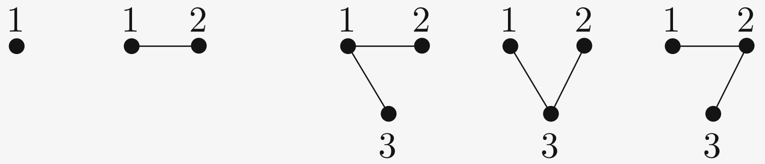

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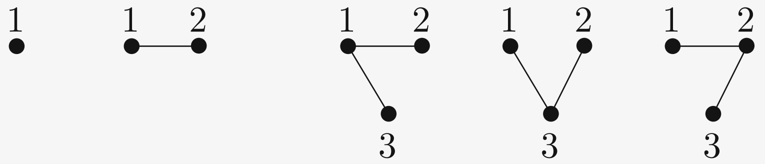
What is the number of spanning trees in a labelled complete graph on n vertices?

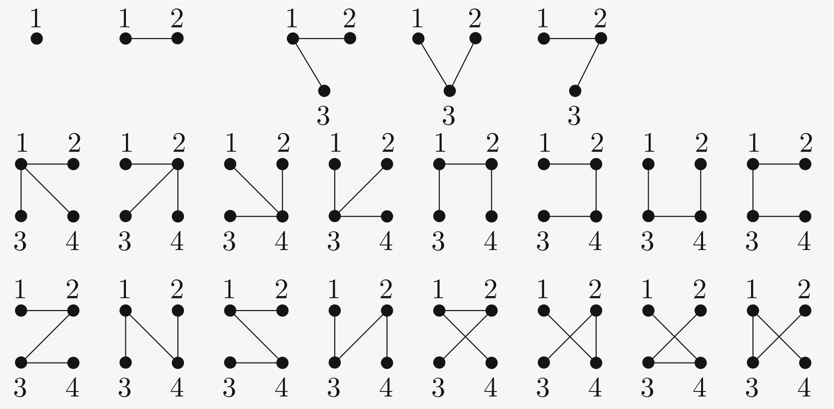


The same question is:

Question

How many different trees can we get on the vertex set $N = \{1, 2, \ldots, n\}$?

Labelled trees of small order



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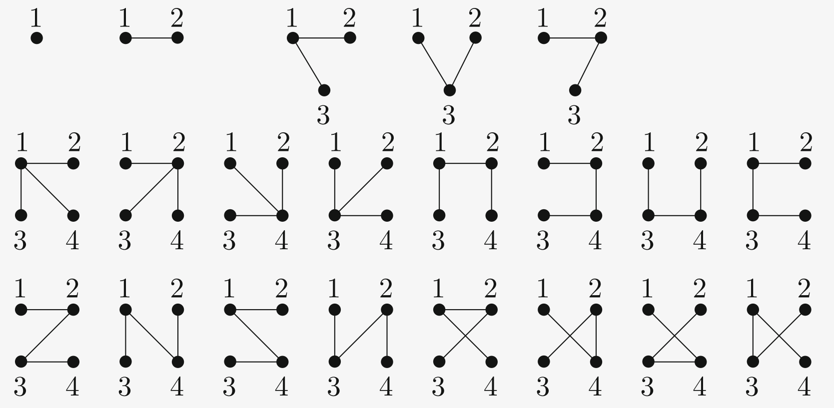
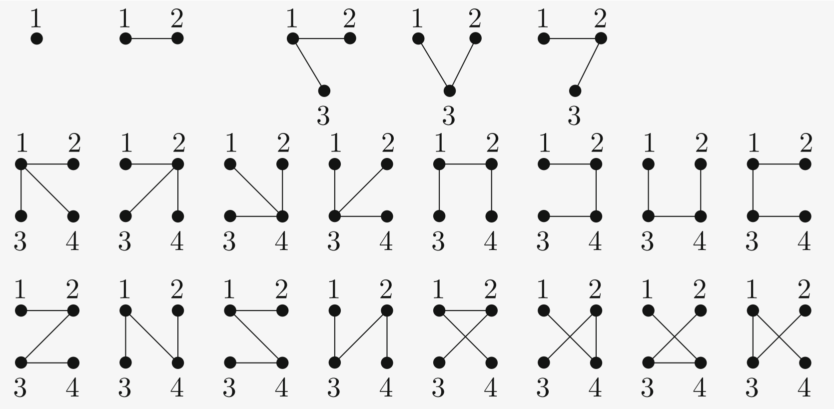
Labelled trees of small order



Enumeration "by hand" yields $T_1 = 1$, $T_2 = 1$, $T_3 = 3$, $T_4 = 16$.

Labelled trees of small order



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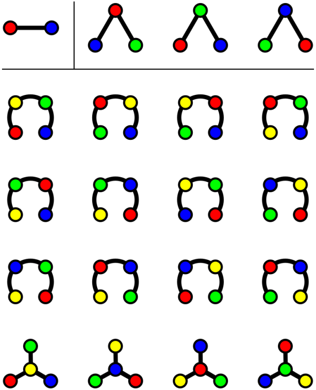
Conjecture that $T_n = n^{n-2}$

Theorem (Cayley's formula)

There are n^{n-2} trees with vertex set $[n]$.



Prüfer code

To prove Cayley's formula, the most direct method is to find a bijection from the set of all trees on n vertices onto another set whose cardinality is known to be n^{n-2}.

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$$(a_1, a_2, \ldots, a_{n-2}) \text{ with } 1 \le a_i \le n$$

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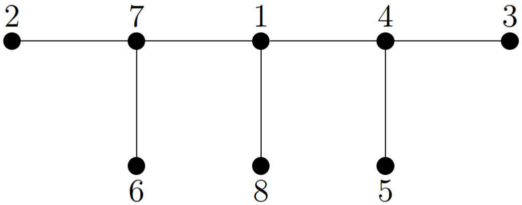
Definition (Prüfer code)

Let T be a tree on an ordered set S of n vertices. To compute the Prüfer sequence $f(T)$, iteratively delete the leaf with the smallest label and append the label of its neighbour to the sequence. After $n-2$ iterations a single edge remains and we have produced a sequence $f(T)$ of length $n-2$.

Get a Prüfer code from a tree

Iteratively delete the leaf with the smallest label and append the label of its neighbour to the sequence.

$T =$



$$f(T) = \emptyset$$

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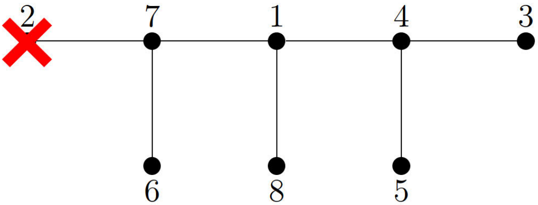

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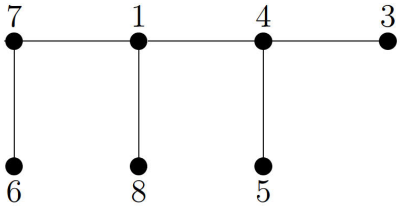

$$f(T) = (7)$$

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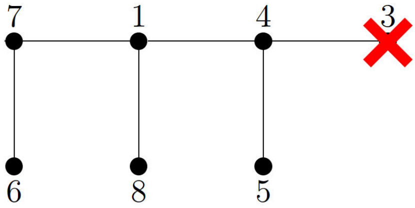

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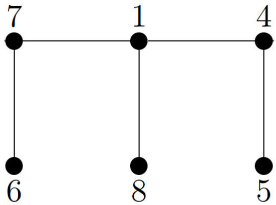

$$f(T) = (7, 4)$$

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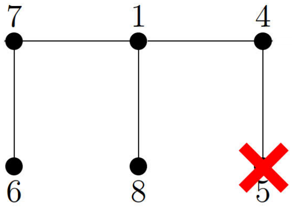

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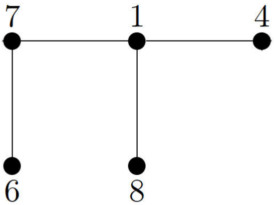

$$f(T) = (7, 4, 4)$$

Get a Prüfer code from a tree

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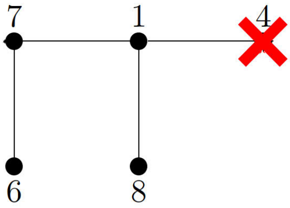

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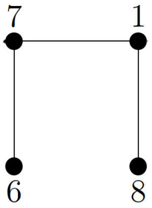
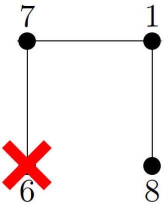


$$f(T) = (7, 4, 4, 1)$$

Iteratively delete the leaf with the smallest label and append the label of its neighbour to the sequence.



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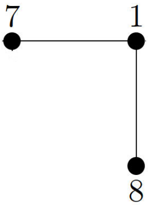

$$f(T) = (7, 4, 4, 1, 7)$$

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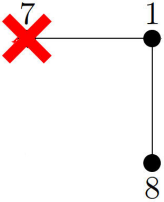

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Get a Prüfer code from a tree

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After $n-2$ iterations a single edge remains and we have produced the sequence $f(T)$ of length $n-2$.

Get a Prüfer code from a tree

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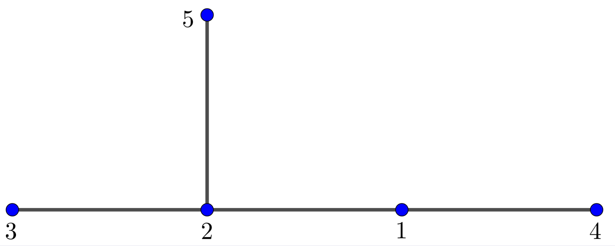
Question

Find the Prüfer sequence $f(T)$ of T, where T is the following tree:



Get a tree from a Prüfer code

Question

Get a labelled tree T from a Prüfer sequence S.

$$S = (a_1, a_2, \ldots, a_{n-2}) \text{ with } 1 \le a_i \le n$$

At each step,

- (a) Find the smallest unmarked element x not in S;
- (b) join x to the first element of S, delete the first element from S and mark x.

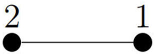
Get a tree from a Prüfer code

Example

Compute the tree with Prüfer code 16631.

$S = 16631$, the unmarked elements are $\{1, 2, 3, 4, 5, 6, 7\}$.

- (a) The smallest unmarked element not in S is 2;
- (b) join 2 to 1, delete 1 from S and mark 2.



Get a tree from a Prüfer code

Example

Compute the tree with Prüfer code 16631.

$S = 6631$, the unmarked elements are $\{1, 3, 4, 5, 6, 7\}$.

- (a) The smallest unmarked element not in S is 4;
- (b) join 4 to 6, delete 6 from S and mark 4.

Get a tree from a Prüfer code

Example

Compute the tree with Prüfer code 16631.

$S = 631$, the unmarked elements are $\{1, 3, 5, 6, 7\}$.

- (a) The smallest unmarked element not in S is 5;
- (b) join 5 to 6, delete 6 from S and mark 5.

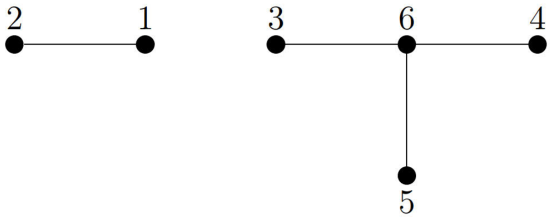
Get a tree from a Prüfer code

Example

Compute the tree with Prüfer code 16631.

$S = 31$, the unmarked elements are $\{1,3,6,7\}$.

- (a) The smallest unmarked element not in S is 6;
- (b) join 6 to 3, delete 3 from S and mark 6.



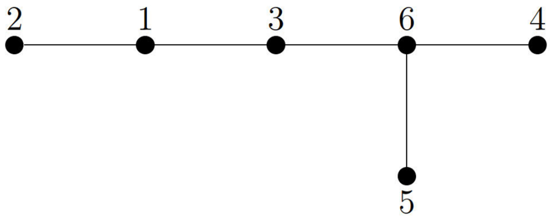
Get a tree from a Prüfer code

Example

Compute the tree with Prüfer code 16631.

$S = 1$, the unmarked elements are $\{1, 3, 7\}$.

- (a) The smallest unmarked element not in S is 3;
- (b) join 3 to 1, delete 1 from S and mark 3.

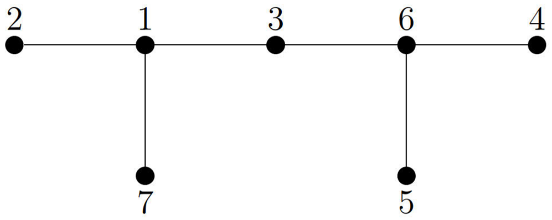


Example

Compute the tree with Prüfer code 16631.

$S = \emptyset$, the unmarked elements are $\{1, 7\}$.

Now add an edge between the remaining vertices $\{1, 7\}$.



At each step,

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Question

A tree T has Prüfer code $f(T) = (1,1,3,1)$. Draw this tree.

At each step,

- (a) Find the smallest unmarked element x not in S;
- (b) join x to the first element of S, delete the first element from S and mark x.

Question

A tree T has Prüfer code $f(T) = (2,2,3)$. Write down all the leaves of this tree.

Proposition

For an ordered n-element set S, the Prüfer code f is a bijection between the trees with vertex set S and the sequences in S^{n-2}.

Bijection between a tree and a Prüfer code

Proposition

For an ordered n-element set S, the Prüfer code f is a bijection between the trees with vertex set S and the sequences in S^{n-2}.

Proof.

We need to show every sequence $(a_1,a_2,\ldots,a_{n-2}) \in S^{n-2}$ defines a unique tree T such that $f(T) = (a_1,a_2,\ldots,a_{n-2})$. If $n = 2$, then there is exactly one tree on 2 vertices and the algorithm defining f always outputs the empty sequence, the only sequence of length zero. So the claim clearly holds for $n = 2$.

Now, assume $n > 2$ and the claim holds for all ordered vertex sets S' of size less than n. Consider a sequence $(a_1,a_2,\ldots,a_{n-2}) \in S^{n-2}$. We need to show that $(a_1,a_2,\ldots,a_{n-2})$ can be uniquely produced by the algorithm.

Bijection between a tree and a Prüfer code

Proposition

For an ordered n-element set S, the Prüfer code f is a bijection between the trees with vertex set S and the sequences in S^{n-2}.

Proof.

(continued) Suppose that the algorithm produces $f(T) = (a_1, a_2, \ldots, a_{n-2})$ for some tree T. Then the vertices $\{a_1, a_2, \ldots, a_{n-2}\}$ are precisely those that are not a leaf in T. Indeed, if a vertex v is a leaf in T then it can only appear in $f(T)$ if its neighbour gets deleted during the algorithm. But this would leave v as an isolated vertex, which is impossible. Conversely, if a vertex v is not a leaf then one of its neighbours must be deleted during the algorithm (it cannot be itself deleted before this happens). When this neighbour of v is deleted, v will be added to the Prüfer code for T, so is in $\{a_1, a_2, \ldots, a_{n-2}\}$.

Proposition

For an ordered n-element set S, the Prüfer code f is a bijection between the trees with vertex set S and the sequences in S^{n-2}.

Proof.

(continued) This implies that the label of the first leaf removed from T is the minimum element of the set $S \setminus \{a_1, a_2, \ldots, a_{n-2}\}$. Let v be this element. In other words, in every T such that $f(T) = (a_1, a_2, \ldots, a_{n-2})$ the vertex v is a leaf whose unique neighbour is a_1.

By induction, there is a unique tree T' with vertex set $S \setminus v$ such that $f(T') = (a_2, \ldots, a_{n-2})$. Adding the vertex v and the edge (a_1, v) to T' yields the desired unique tree T with $f(T) = (a_1, a_2, \ldots, a_{n-2})$. □

Thank you!