Lecture 3. Graph parameters and trees

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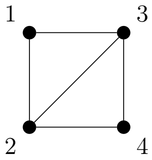
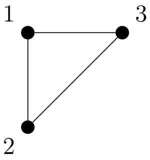
Clique, independent set

Definition

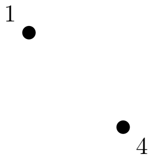
A clique in G is a complete subgraph in G. An independent set is an empty induced subgraph in G.

$G =$





clique in G



independent set in G

Clique number and independence number

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Let $\omega(G)$ denote the number of vertices in a maximum-size clique in G; let $\alpha(G)$ denote the number of vertices in a maximum-size independent set in G.

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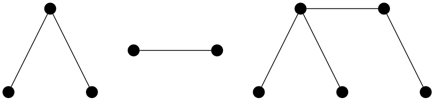
Corollary

We have $\omega(G) = \alpha(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$.

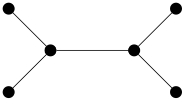
Trees

Definition

A graph having no cycle is acyclic. A forest is an acyclic graph; a tree is a connected acyclic graph. A leaf (or pendant vertex) is a vertex of degree 1.



forest



tree

Leaves of a tree

Lemma

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Suppose v is a leaf of a tree G, and let $G' = G \setminus v$. If $u, w \in V(G')$, then no u,w-path P in G can pass through the vertex v of degree 1, so P is also present in G'. Hence G' is connected. Since deleting a vertex cannot create a cycle, G' is also acyclic. We conclude that G' is a tree with $n-1$ vertices. □

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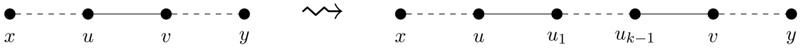
Let (u,v) belong to a cycle. Then any path $x \dots y$ in G which uses the edge (u,v) can be extended to a walk in $G \setminus (u,v)$. □



Equivalent definitions of trees

Theorem

For an n-vertex simple graph G (with $n \geq 1$), the following are equivalent (and characterize the trees with n vertices).

- *(a) G is connected and has no cycles.*
- *(b) G is connected and has $n-1$ edges.*
- *(c) G has $n-1$ edges and no cycles.*
- *(d) For every pair $u, v \in V(G)$, there is exactly one (u,v)-path in G.*

Proof.

$(a) \Rightarrow (b),(c)$: We use induction on n. For $n = 1$, an acyclic 1-vertex graph has no edge. For the induction step, suppose $n > 1$, and suppose the implication holds for graphs with fewer than n vertices. Given G, we can find a leaf v such that $G' = G \setminus v$ is acyclic and connected. Applying the induction hypothesis to G' yields $e(G') = n - 2$, and hence $e(G) = n - 1$.

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$(b) \Rightarrow (a),(c)$: Delete edges from cycles of G one by one until the resulting graph G' is acyclic. By the last lemma, G' is connected. By the above paragraph, G' has $n - 1$ edges. Since this equals $|E(G)|$, no edges were deleted, and G itself is acyclic. □

Proof.

$(c) \Rightarrow (a),(b)$: Suppose G has k components with orders $n_1, \ldots, n_k$. Since G has no cycles, each component satisfies property (a), and by the first paragraph the ith component has $n_i - 1$ edges. Summing this over all components yields $e(G) = \sum(n_i - 1) = n - k$. We are given $e(G) = n - 1$, so $k = 1$, and G is connected.

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$(d) \Rightarrow (a)$: If there is a u,v-path for every $u,v \in V(G)$, then G is connected. If G has a cycle C, then G has two paths between any pair of vertices on C. □

Proof.

$(a) \Rightarrow (d)$: Since G is connected, G has at least one u,v-path for each pair $u,v \in V(G)$. Suppose G has distinct u,v-paths P and Q. Let $e = (x,y)$ be an edge in P but not in Q. The concatenation of P with the reverse of Q is a closed walk in which e appears exactly once. Hence, $(P \cup Q) \setminus e$ is an x,y-walk not containing e. This x,y-walk contains an x,y-path, which completes a cycle with e and contradicts the hypothesis that G is acyclic. Hence G has exactly one u,v-path. □

Spanning tree

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Corollary

- *(a) Every connected graph on n vertices has at least $n-1$ edges and contains a spanning tree;*
- *(b) Every edge of a tree is a cut-edge;*
- *(c) Adding an edge to a tree creates exactly one cycle.*

Proof.

- (a) Delete edges from cycles of G one by one until the resulting graph G' is acyclic. Because G' is connected, it is a tree. Therefore G contains a spanning tree and has at least $n-1$ edges.
- (b) Note that deleting an edge from a tree T on n vertices leaves $n-2$ edges, so the graph is disconnected by (a).
- (c) Let $u, v \in T$. There is a unique path in T between u and v, so adding an edge (u, v) closes this path to a unique cycle.

□

Thank you!