Lecture 2. Basic notions (2)

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Subgraph, spanning subgraph, induced subgraph

Definition

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Given $G = (V, E)$ and $U \subseteq V$ ($U \neq \emptyset$), let $G[U]$ denote the graph with vertex set U and edge set $E(G[U]) = \{e \in E(G) \mid e \subseteq U\}$. (We include all the edges of G which have both endpoints in U). Then $G[U]$ is called the subgraph of G induced by U.

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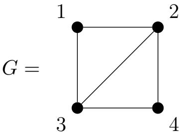
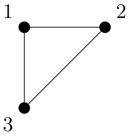
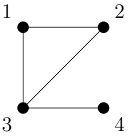




induced subgraph



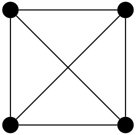
not induced
but spanning

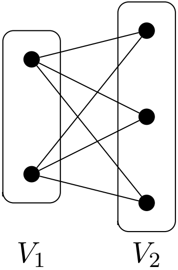
Special graphs

Definition

- K_n is the complete graph, or a clique. Take n vertices and all possible edges connecting them.
- An empty graph has no edges.
- $G = (V, E)$ is bipartite if there is a partition $V = V_1 \cup V_2$ into two disjoint sets such that each $e \in E(G)$ intersects both V_1 and V_2.
- $K_{n,m}$ is the complete bipartite graph. Take $n + m$ vertices partitioned into a set A of size n and a set B of size m, and include every possible edge between A and B.

Example

$K_4 =$ 

$K_{2,3} =$ 

Definition

A walk in G is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$, and a sequence of edges $(v_i, v_{i+1}) \in E(G)$. A walk is a path if all v_i are distinct. If for such a path with $k \geq 2$, (v_0, v_k) is also an edge in G, then $v_0, v_1, \ldots, v_k, v_0$ is a cycle. For multigraphs, we also consider loops and pairs of multiple edges to be cycles.

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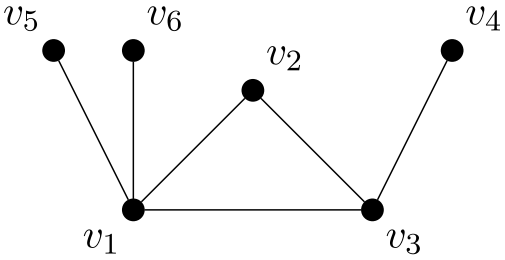
Definition

The length of a path, cycle or walk is the number of edges in it.

Example



$v_5v_1v_3v_4$ is a path of length 3;

$v_1v_2v_3v_1$ is a cycle of length 3;

$v_5v_1v_2v_3v_1v_6$ is a walk of length 5.

Proposition

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Every walk from u to v in G contains a path between u and v.

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Proof.

By induction on the length ℓ of the walk $u = u_0, u_1, \ldots, u_\ell = v$.
If $\ell = 1$ then our walk is also a path. Otherwise, if our walk is not a path there is $u_i = u_j$ with $i < j$, then $u = u_0, \ldots, u_i, u_{j+1}, \ldots, v$ is also a walk from u to v which is shorter. We can use induction to conclude the proof. □

Proposition

Every G with minimum degree $\delta \geq 2$ contains a path of length δ and a cycle of length at least $\delta + 1$.

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Proof.

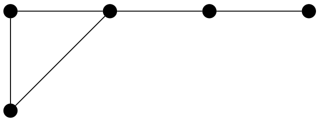
Let $v_1, \ldots, v_k$ be a longest path in G. Then all neighbours of v_k belong to $v_1, \ldots, v_{k-1}$ so $k - 1 \geq \delta$ and $k \geq \delta + 1$, and our path has at least δ edges. Let i $(1 \leq i \leq k-1)$ be the minimum index such that $(v_i, v_k) \in E(G)$. Then the neighbours of v_k are among $v_i, \ldots, v_{k-1}$, so $k - i \geq \delta$. Then $v_i, v_{i+1}, \ldots, v_k$ is a cycle of length at least $\delta + 1$. □

Remark

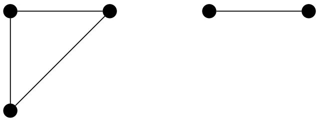
Note that we have also proved that a graph with minimum degree $\delta \geq 2$ contains cycles of at least $\delta - 1$ different lengths. This fact, and the statement of the proposition, are both tight; to see this, consider the complete graph $G = K_{\delta+1}$.

Definition

A graph G is connected if for all pairs $u, v \in G$, there is a path in G from u to v.



connected



not connected

Connected component

Definition

A (connected) component of G is a connected subgraph that is maximal by inclusion. We say G is connected if and only if it has one connected component.

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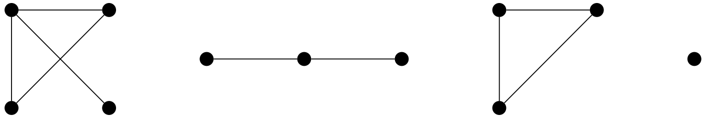
$G =$



G has 4 connected components.

Proposition

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A graph with n vertices and m edges has at least $n - m$ connected components.

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Proof.

Start with the empty graph (which has n components), and add edges one-by-one. Note that adding an edge can decrease the number of components by at most 1. □

Complement graph

Definition

Given $G = (V, E)$, the complement $\overline{G}$ of G has the same vertex set V and $(u,v) \in E(\overline{G})$ if and only if $(u,v) \notin E(G)$.

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Thank you!