Topics in Graph Theory

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About the literature

We will focus on the book Graph Theory written by Benny Sudakov. We study only Chapters 1–7 (Pages 1–45).

It has been used as a textbook by several top universities like Swiss Federal Institute of Technology Zurich (ETH), Massachusetts Institute of Technology (MIT).

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Recommended Literature:

- Douglas B. West, Introduction to Graph Theory, Second Edition, 2001.
- Douglas B. West, Introduction to Graph Theory, Solution Manual, 2005.

1. Lecture 1. Basic notions (1)
   - Graphs
   - Graph isomorphism
   - The adjacency and incidence matrices
   - Degree

What is a graph?

Definition

A graph G is a pair $G = (V, E)$ where V is a set of vertices and E is a (multi)set of unordered pairs of vertices. The elements of E are called edges. We write $V(G)$ for the set of vertices and $E(G)$ for the set of edges of a graph G. Also, $|G| = |V(G)|$ denotes the number of vertices and $e(G) = |E(G)|$ denotes the number of edges.

Simple graph

Definition

A loop is an edge (v,v) for some $v \in V$. An edge $e = (u,v)$ is a multiple edge if it appears multiple times in E. A graph is simple if it has no loops or multiple edges.

Unless explicitly stated otherwise, we will only consider simple graphs. General (potentially nonsimple) graphs are also called multigraphs.

Relations

Definition

If $e = (u, v) \in E(G)$, then

- u, v are adjacent (neighbours);
- e is incident to u, v;
- e joins u and v;
- u and v are the ends of e.

Two edges e, e' are adjacent if $e \cap e' \neq \emptyset$.

Real world applications

- Let V be the set of people in the room, and let E be the set of pairs of people who met for the first time today.
- Let V be the set of cities in a country, and let the edges in E correspond to roads connecting them.

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Problem (American Mathematical Monthly, 1959)

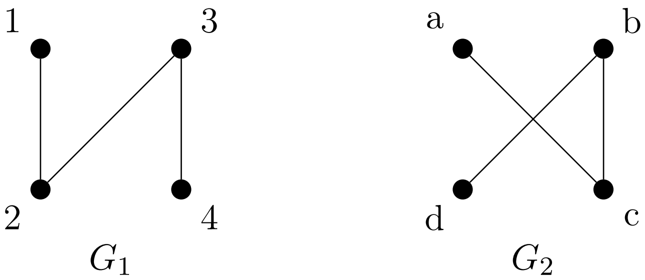
Prove that, at any party of six people, we can find three mutual acquaintances (each one knows the other two) or three mutual strangers (each one does not know the other two).

Graph isomorphism

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An isomorphism $\phi : G_1 \to G_2$ is a bijection (a one-to-one correspondence) from V_1 to V_2 such that $(u, v) \in E_1$ if and only if $(\phi(u), \phi(v)) \in E_2$. We say G_1 is isomorphic to G_2 if there is an isomorphism between them.

Example



The function $\phi : G_1 \to G_2$ given by $\phi(1) = a$, $\phi(2) = c$, $\phi(3) = b$, $\phi(4) = d$ is an isomorphism.

Remark

Isomorphism is an equivalence relation of graphs. This means that

- any graph is isomorphic to itself;
- if G_1 is isomorphic to G_2, then G_2 is isomorphic to G_1;
- if G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3.

Unlabelled graph

Definition

An unlabelled graph is an isomorphism class of graphs. In the previous example G_1 and G_2 are different labelled graphs but since they are isomorphic they are the same unlabelled graph.

The adjacency matrix

Let $[n] = \{1, \ldots, n\}$.

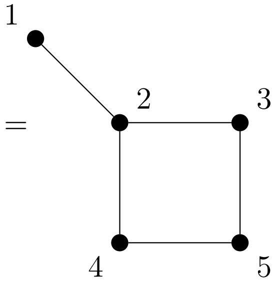
Definition

Let $G = (V, E)$ be a graph with $V = [n]$. The adjacency matrix $A = A(G)$ is the $n \times n$ symmetric matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Example

$G =$



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The incidence matrix

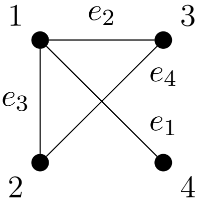
Definition

Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. Then the incidence matrix $B = B(G)$ of G is the $n \times m$ matrix defined by

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise.} \end{cases}$$

Example

$G =$



$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Degree

Definition

Given $G = (V, E)$ and a vertex $v \in V$, we define the neighbourhood $N(v)$ of v to be the set of neighbours of v. Let the degree $d(v)$ of v be $|N(v)|$, the number of neighbours of v. A vertex v is isolated if $d(v) = 0$.

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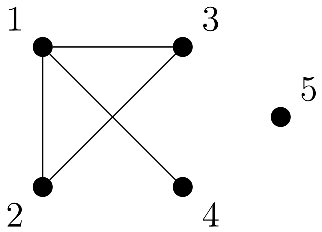
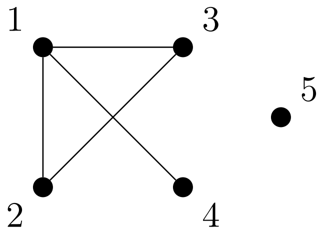
Remark. $d(v)$ is the number of 1s in the row corresponding to v in the adjacency matrix $A(G)$ or the incidence matrix $B(G)$.

Example



Example



$d(1) = 3$, $d(2) = 2$, $d(3) = 2$, $d(4) = 1$, $d(5) = 0$;

5 is isolated.

Fact

Fact

For any graph G on the vertex set $[n]$ with adjacency and incidence matrices A and B, we have $BB^T = D + A$, where

$$D = \begin{pmatrix} d(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d(n) \end{pmatrix}.$$

Degree notation

Definition

The minimum degree of a graph G is denoted $\delta(G)$; the maximum degree is denoted $\Delta(G)$. The average degree is

$$\overline{d}(G) = \frac{\sum_{v \in G} d(v)}{|V(G)|}$$

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Note that $\delta(G) \le \overline{d}(G) \le \Delta(G)$.

Regular graph

Definition

A graph G is d-regular if and only if all vertices have degree d.

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Is there a 3-regular graph on 9 vertices?

Handshaking lemma

Theorem (Degree sum formula)

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Proof.

In the sum $\sum_{v \in G} d(v)$, every edge $e = (u, v)$ is counted twice: once from u and once from v. □

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Corollary (Handshaking lemma)

Every graph has an even number of vertices of odd degree.

Thank you!